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Harmonic Conjugates in Weighted Bergman Spaces of Quaternion-Valued Functions

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Abstract. In this paper we study the harmonic conjugation problem in weighted Bergman spaces of quaternion-valued functions on the unit ball. For a scalar-valued harmonic function belonging to a Bergman space, harmonic conjugates in the same Bergman space are found.

Keywords. Quaternion, monogenic function, harmonic Bergman spaces.

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1. Introduction and notation

Hardy and Littlewood [17, Thm. 5] were the first who considered the problem of harmonic conjugation in Bergman spaces on the unit disk of the complex plane. Among many generalizations, an important system of harmonic conjugates in $\mathbb{R}^n$ was introduced by Stein and Weiss (see [27]), which played a crucial role in the characterization of Hardy spaces on the upper half-space $\mathbb{R}^{n+1}_+$. The problem of harmonic conjugates in the framework of Clifford analysis was already studied by several authors. In 1979, for the case of a harmonic function in $\mathbb{R}^4$ an explicit formula for conjugate harmonic functions is given by Sudbery [30, Thm. 4] such that a quaternion-valued monogenic function is defined. For arbitrary dimensions we find a generalization of the mentioned formula in [7]. Some papers are dealing with the problem by constructing harmonic conjugates to the Poisson kernel (see [5, 11, 31]). Another approach is to relate the problem of harmonic conjugates to singular integral equations as it was done, e.g. in [9, 25]. More general questions like uniqueness (under some conditions), existence and the construction of harmonic conjugates for special functions (for instance polynomials) play a role in [6, 8, 10]. This list does not claim completeness but shows that the topic of harmonic conjugates is relatively well studied in the framework of Clifford analysis, too. In all these papers one question is not answered — if

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the given harmonic function belongs to a certain function space, where are then the conjugate harmonic functions and so constructed monogenic function?

The purpose of the present paper is to study the harmonic conjugation in weighted Bergman spaces in the framework of quaternionic analysis. We shall be mainly using Sudbery's formula (see [30]) for the construction of harmonic conjugates and the study of their properties. Several times we will work with known results from classical harmonic analysis. We will refer to them in their general $\mathbb{R}^n$-form even though they are applied only for $n = 4$.

Let $n \geq 2$ be an integer, and $B = B_n$ be the open unit ball in the n-dimensional Euclidean space $\mathbb{R}^n$, and $S = \partial B$ be its boundary, the unit sphere. Besides the general space $\mathbb{R}^n$, we will work in $\mathbb{H} \cong \mathbb{R}^4$, the skew field of real quaternions. Each element of $\mathbb{H}$ can be written in the form

$$x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}, \qquad x_0, x_1, x_2, x_3 \in \mathbb{R}$$

where the system $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ forms a basis of $\mathbb{H}$, and

$$\mathbf{Sc}\, x = x_0, \qquad \mathbf{Vec}\, x = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}.$$

The corresponding multiplication rules are given by

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \qquad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \qquad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j},$$

The conjugate element to $x \in \mathbb{H}$ is defined by

$$\bar{x} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k},$$

and so

$$x\bar{x} = \bar{x}x = |x|^2 = {x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2.$$

As usually, let $\mathbb{Z}_0$ denote the set of all non-negative integers. For a multi-index $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}_0^4$ let $\partial^\lambda = \partial_x^\lambda$ denote the partial differential operator of the order $|\lambda| = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3$ with respect to x_0, x_1, x_2, x_3.

Let D denote the Cauchy-Riemann-Fueter operator

$$D = \frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2} + \mathbf{k}\frac{\partial}{\partial x_3} = \partial_0 + \mathbf{i}\partial_1 + \mathbf{j}\partial_2 + \mathbf{k}\partial_3,$$

and $\overline{D}$ its conjugate operator

$$\overline{D} = \frac{\partial}{\partial x_0} - \mathbf{i}\frac{\partial}{\partial x_1} - \mathbf{j}\frac{\partial}{\partial x_2} - \mathbf{k}\frac{\partial}{\partial x_3} = \partial_0 - \mathbf{i}\partial_1 - \mathbf{j}\partial_2 - \mathbf{k}\partial_3.$$

A real-differentiable function $f = u_0 + u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, is said to be (left) monogenic if $Df = 0$. We refer to [7, 15, 16] for the general theory of quaternionic and Clifford analysis. We denote by $\mathcal{M}(B_4, \mathbb{H})$, $h(B_4, \mathbb{H})$, $h(B_n, \mathbb{R})$, $h(B_n, \mathbb{C})$ the sets of monogenic, quaternion-valued harmonic, real-valued harmonic and complex-valued harmonic functions, respectively, given in the unit ball.

Throughout the paper, the letters $C(\alpha, \beta, \dots)$, C_α, etc. stand for positive constants possibly different at different places and depending only on the parameters indicated. For $A, B > 0$ the notation $A \approx B$ denotes the two-sided estimate

$c_1 A \leq B \leq c_2 A$ with some positive constants c_1 and c_2 independent of the variables involved.

2. Monogenic Bergman spaces

For a real- or vector-valued function $f(x) = f(r\zeta)$ in B_n, $0 \leq r < 1, \zeta \in S$, its integral mean is defined by

$$M_p(f;r) = \|f(r\cdot)\|_{L^p(S,d\sigma)}, \qquad 0 \leq r < 1,\, 0 < p \leq \infty,$$

where $d\sigma$ is the normalized surface measure on S. The Bergman norm of a measurable function in B_n (or a quaternion-valued function in B_4) is defined by

$$\|f\|_{p,\alpha} = \left(\int_{B_n} (1-|x|)^\alpha |f(x)|^p)\, dV_n(x)\right)^{1/p}, \qquad 0 < p < \infty,\, \alpha > -1,$$

where dV_n is the Lebesgue volume measure on B_n normalized by $V_n(B_n) = 1$. In the polar coordinates, we have

$$dV_n(x) = nr^{n-1}\, dr\, d\sigma(\zeta),$$

[3, p. 6]. Corresponding weighted Bergman spaces $\mathcal{M}^p_\alpha$ of monogenic functions in B_4 and spaces h^p_α and $\mathbf{h}^p_\alpha$ of (scalar- or quaternion-valued) harmonic functions are defined by

$$\begin{aligned}
\mathcal{M}^p_\alpha &= \{f \in \mathcal{M}(B_4, \mathbb{H})\colon \|f\|_{p,\alpha} < +\infty\}, \\
h^p_\alpha &= \{u \in h(B_n, \mathbb{R}) \text{ or } u \in h(B_n, \mathbb{C})\colon \|u\|_{p,\alpha} < +\infty\}, \\
\mathbf{h}^p_\alpha &= \{u \in h(B_4, \mathbb{H})\colon \|u\|_{p,\alpha} < +\infty\}.
\end{aligned}$$

For the basic theory of harmonic Bergman spaces, see [3, 18, 19]. For Bergman and other closely related spaces of Clifford valued functions in B_n, see also [4].

It is well known that the Cauchy kernel

$$e(x) = \frac{1}{\sigma_3} \frac{\bar{x}}{|x|^4}$$

is monogenic in $\mathbb{R}^4 \setminus \{0\}$, where σ_3 is the surface-area of the unit sphere S_3 in $\mathbb{R}^4$. We will consider its modified translation

$$E(x,y) = e(\rho x - \xi), \tag{1}$$

where $x = r\zeta$, $y = \rho\xi \in B_4$, $\zeta, \xi \in S$, $0 \leq r < 1$, $0 \leq \rho \leq 1$.

For a function f monogenic in a bounded domain Ω and continuous in the closure of Ω, the Cauchy integral formula holds

$$f(x) = \int_{\partial\Omega} e(x-\xi) n(\xi) f(\xi)\, ds(\xi), \qquad x \in \Omega, \tag{2}$$

where $n(\xi)$ is the outward normal unit vector to $\partial\Omega$ at the point ξ, and ds is the surface-area measure on $\partial\Omega$.

The two surface measures $d\sigma$ and ds are related by the equality $d\sigma = ds/\sigma_3$. We will use both notations in parallel to avoid misunderstandings in citations. The normalized measure $d\sigma$ is mostly used in the definitions of integral means, Bergman kernels and norms. For the definition of integral operators (in our case the Cauchy integral operator) the measure ds is preferred because the normalization constant $1/\sigma_3$ is already contained in the definition of the Cauchy kernel (fundamental solution of D).

Lemma 1. *Let $\beta > \alpha > -1$. Then for any $x = r\zeta \in B_n$*

$$\int_S \frac{d\sigma(\xi)}{|\xi - x|^{\beta+n}} \le C(\beta, n)\, \frac{1}{(1-|x|)^{\beta+1}}, \tag{3}$$

$$\int_{B_n} \frac{(1-|y|)^{\alpha}}{|\zeta - ry|^{\beta+n}}\, dV_n(y) \le C(\alpha, \beta, n)\, \frac{1}{(1-|x|)^{\beta-\alpha}}. \tag{4}$$

The estimates of Lemma 1 are well known and can be found, for example, in [18, pp. 87–88], [19, pp. 29–30], [21, p. 90].

Lemma 2. *For any multi-index $\lambda \in \mathbb{Z}_0^4$ and $3/(3+|\lambda|) < p \le \infty$, we have*

$$|\partial^{\lambda} e(x)| \le C_{\lambda}\, \frac{1}{|x|^{3+|\lambda|}}, \qquad x \in B_4, \tag{5}$$

$$M_p(\partial_x^{\lambda} E; r) \le C(\lambda, p) \frac{1}{(1-r)^{3+|\lambda|-3/p}}, \qquad 0 \le r < 1. \tag{6}$$

These estimates for the Cauchy kernel are known; the estimate (5) is elementary, and the estimate (6) follows from (1), (5) and (3).

We also need the next well-known lemma, see, e.g. [27], [26, Lem. 8].

Lemma 3 (Hardy and Hardy type inequalities)**.** (i) *If $1 \le p < \infty$, $\beta > -1$, $g(r) \ge 0$, then*

$$\int_0^1 (1-r)^{\beta} \left(\int_0^r g(t)\, dt \right)^p dr \le C \int_0^1 (1-r)^{\beta+p} g^p(r)\, dr,$$

where the constant C depends only on β, p.

(ii) *If $0 < p < 1$ and $g(r)$ is a positive increasing function, then*

$$\left(\int_0^1 g(tr)\, dt \right)^p \le C_p \int_0^1 (1-t)^{p-1} g^p(tr)\, dt, \qquad 0 \le r < 1.$$

Lemma 4. *Suppose $0 < p < \infty$, $\alpha > -1$. Then for all $u \in h(B_n)$*

$$\int_{B_n} (1-|x|)^{\alpha} |u(x)|^p\, dV_n(x) \approx \int_0^1 (1-r)^{\alpha} M_p^p(u; r)\, dr.$$

Proof. For $p \ge 1$ the result is clear because of subharmonicity of $|u|^p$ and monotonicity of the integral means $M_p(u; r)$ with regard to r. So, we need to prove the lemma only for $0 < p < 1$. The proof below is, however, true for all $0 < p < \infty$. It suffices to prove the inequality

$$\int_0^{1/2} (1-r)^\alpha M_p^p(u;r)\,dr \le C(p,\alpha,n) \int_0^1 (1-r)^\alpha M_p^p(u;r) r^{n-1}\,dr. \tag{7}$$

For any point x with $|x| < 1/2$, take the ball

$$B(x) = \left\{ y \in B_n \colon |y-x| < \frac{1}{2}(1-|x|) \right\}$$

and write the Hardy-Littlewood-Fefferman-Stein inequality [14, p. 172] (often called HL-property for $|u|^p$)

$$|u(x)|^p \le \frac{C(p,n)}{(1-|x|)^n} \int_{B(x)} |u(y)|^p \,dV_n(y).$$

Since $1-|y| \approx 1-|x|$ for $y \in B(x)$, and $B(x) \subset \{y \colon |y| < 3/4\}$, we get

$$\begin{aligned} |u(x)|^p &\le \frac{C(p,\alpha,n)}{(1-|x|)^{n+\alpha}} \int_{B(x)} (1-|y|)^\alpha |u(y)|^p \,dV_n(y) \\ &\le C(p,\alpha,n) 2^{n+\alpha} \int_{|y|<3/4} (1-|y|)^\alpha |u(y)|^p \,dV_n(y). \end{aligned}$$

It follows that

$$\begin{aligned} \int_0^{1/2} \int_S (1-r)^\alpha |u(r\zeta)|^p \,dr\,d\sigma(\zeta) &\le C(\alpha,n) \sup_{|x|<1/2} |u(x)|^p \\ &\le C(p,\alpha,n) \int_0^{3/4} (1-r)^\alpha M_p^p(u;r) r^{n-1}\,dr. \end{aligned}$$

Thus, we have even obtained a stronger inequality than (7). ■

3. Differentiation in $\mathcal{M}_\alpha^p$ and $\mathbf{h}_\alpha^p$

There are many well-known theorems in Bergman and more general weighted function spaces which relate the norm of the function with norms of derivatives in suitable other spaces of the scales, see, for example [1, 12, 13, 20, 23, 26, 28, 29]. So, the next lemma is essentially known. In the context of Clifford analysis such a result can be found in [23].

Lemma 5. *Let $1 \le p < \infty$, $\alpha > -1$, m be a positive integer, and $\lambda \in \mathbb{Z}_0^4$. Then for all quaternion-valued monogenic or harmonic functions f the following relations hold:*

$$\|f\|_{p,\alpha} \approx \sum_{|\lambda|<m} \left|\partial^\lambda f(0)\right| + \sum_{|\lambda|=m} \left\|\partial^\lambda f\right\|_{p,\alpha+pm}, \tag{8}$$

$$\|f\|_{p,\alpha} \approx |f(0)| + \|\nabla f\|_{p,\alpha+p}, \tag{9}$$

where ∇ stands for the gradient. The involved constants depend only on p, α, m.

Proof. For a monogenic function $f(x)=f(r\zeta)\in\mathcal{M}^p_\alpha$ we apply the Cauchy integral formula (2) with respect to the dilated function f_δ defined by $f_\delta(x)=f(\delta x)$ and obtain

$$f(\delta x)=\int_S E(x,\xi)\,n(\xi)\,f(\delta\xi)\,ds(\xi),\qquad x=r\zeta\in B_4,\ 0<\delta<1,$$

where $E(x,\xi)=e(x-\xi)$ is the kernel (1). (For quaternion-valued harmonic functions we use Poisson integral formula instead of the Cauchy formula.) Take then the partial differential operator ∂^λ_x

$$\partial^\lambda_x f(\delta x)=\int_S \partial^\lambda_x E(x,\xi)\,n(\xi)\,f(\delta\xi)\,ds(\xi),$$

and estimate by Lemma 2

$$\left|\partial^\lambda f(\delta x)\right|\le C_\lambda\int_S\frac{|f(\delta\xi)|}{|x-\xi|^{3+|\lambda|}}\,d\sigma(\xi). \tag{10}$$

Replace x in (10) by Tx, where T is an arbitrary orthogonal linear transformation $T\colon\mathbb{R}^4\to\mathbb{R}^4$, that is, $|Tx|=|x|$ for all $x\in\mathbb{R}^4$. Recall that the measure σ is invariant under rotations, meaning $\sigma\big(T(G)\big)=\sigma(G)$ for every Borel set $G\subset S$ and every orthogonal transformation T. Applying the change $\xi\mapsto T\xi$ in (10), we find that

$$\left|\partial^\lambda f(\delta Tx)\right|\le C_\lambda\int_S\frac{|f(\delta T\xi)|}{|x-\xi|^{3+|\lambda|}}\,d\sigma(\xi).$$

By Minkowski's inequality and Lemma 1

$$M_p\big(\partial^\lambda f;\delta r\big)\le C_\lambda M_p(f;\delta)\int_S\frac{d\sigma(\xi)}{|x-\xi|^{3+|\lambda|}}\le C_\lambda\frac{M_p(f;\delta)}{(1-r)^{|\lambda|}},$$

where we have also used the identity

$$M_p(F;|z|)=\left(\int|F(Tz)|^p dT\right)^{1/p},\qquad z\in B_4,$$

the integral is taken over the orthogonal group. Putting $\delta=r$

$$(1-r)^{\alpha+p|\lambda|}M_p^p\left(\partial^\lambda f;r^2\right)\le C(1-r)^\alpha M_p^p(f;r),\qquad 0<r<1,$$

and then integrating over the interval $(0,1)$, we obtain

$$\left\|\partial^\lambda f\right\|_{p,\alpha+p|\lambda|}\le C\|f\|_{p,\alpha}$$

for every multi-index $\lambda\in\mathbb{Z}^4_0$. On the other hand, by the subharmonicity of $|\partial^\lambda f(x)|^p$,

$$\left|\partial^\lambda f(x)\right|^p\le\frac{C}{(1-|x|)^{4+p|\lambda|}}\int_{|y-x|<(1-|x|)/2}|f(y)|^p\,dV_4(y),$$

see, e.g. [3, Ch. 8]. Then taking $x=0$ in the inequality

$$\left|\partial^\lambda f(x)\right|^p\le\frac{C}{(1-|x|)^{4+p|\lambda|+\alpha}}\int_{|y-x|<(1-|x|)/2}|f(y)|^p(1-|y|)^\alpha\,dV_4(y),$$

we get

$$\left|\partial^\lambda f(0)\right|^p \le C \int_{B_4} |f(y)|^p (1-|y|)^\alpha \, dV_4(y).$$

Thus,

$$\sum_{|\lambda|<m} \left|\partial^\lambda f(0)\right|^p + \sum_{|\lambda|=m} \left\|\partial^\lambda f\right\|^p_{p,\alpha+pm} \le C \|f\|^p_{p,\alpha}.$$

Conversely, we have

$$f(x) = f(0) + \int_0^r \frac{\partial f(t\zeta)}{\partial t} \, dt = f(0) + \int_0^r \nabla f(t\zeta) \cdot \zeta \, dt,$$

where the dot means the inner product in $\mathbb{R}^4$. Hence by Minkowski's inequality

$$M_p(f;r) \le |f(0)| + \int_0^r M_p(\nabla f; t) \, dt.$$

Then Lemma 3 (i) yields

$$\begin{aligned}\int_0^1 (1-r)^\alpha M_p{}^p(f;r) \, dr &\le C|f(0)|^p + C \int_0^1 (1-r)^\alpha \left(\int_0^r M_p(\nabla f; t) \, dt \right)^p dr \\ &\le C|f(0)|^p + C \int_0^1 (1-r)^{p+\alpha} M_p{}^p(\nabla f; r) \, dr.\end{aligned}$$

Since

$$M_p{}^p(\nabla f; r) \le C \sum_{j=0}^{3} M_p{}^p \left(\frac{\partial f}{\partial x_j}; r \right),$$

we have

$$\int_0^1 (1-r)^\alpha M_p{}^p(f;r) \, dr \le C|f(0)|^p + C \sum_{j=0}^{3} \int_0^1 (1-r)^{p+\alpha} M_p{}^p \left(\frac{\partial f}{\partial x_j}; r \right) dr.$$

Thus,

$$\|f\|_{p,\alpha} \le C|f(0)| + C \sum_{j=0}^{3} \left\| \frac{\partial f}{\partial x_j} \right\|_{p,\alpha+p}. \tag{11}$$

So, the desired inequality is obtained for $m = 1$. Then we apply (11) with respect to $\partial f/\partial x_j$, $j = 0, 1, 2, 3$, and get

$$\left\| \frac{\partial f}{\partial x_j} \right\|_{p,\alpha+p} \le C \left| \frac{\partial f(0)}{\partial x_j} \right| + C \sum_{k=0}^{3} \left\| \frac{\partial^2 f}{\partial x_j \partial x_k} \right\|_{p,\alpha+2p}, \qquad j = 0, 1, 2, 3.$$

Inserting this in (11), we obtain

$$\|f\|_{p,\alpha} \le C \sum_{|\lambda|<2} \left|\partial^\lambda f(0)\right| + C \sum_{|\lambda|=2} \left\| \frac{\partial^2 f}{\partial x_j \partial x_k} \right\|_{p,\alpha+2p}.$$

We complete the proof by induction.

The relation (9) follows easily from (8) with $m = 1$. ∎

4. Harmonic conjugation, case $1 \le p < \infty$

Theorem 1. *Let $u(x) = u_0(x)\colon B_4 \to \mathbb{R}$ be a real-valued harmonic function in the unit ball B_4. If $u \in h^p_\alpha$ for some $\alpha > -1$ and $1 \le p < \infty$, then there exists a monogenic function $f(x) : B_4 \to \mathbb{H}$ such that $f \in \mathcal{M}^p_\alpha$ and* $\mathbf{Sc}\, f = u$ *in B_4, and*

$$\|f\|_{p,\alpha} \le C(p,\alpha)\|u\|_{p,\alpha}.$$

Proof. Sudbery's explicit formula (see [16, p. 42], [30, p. 212]) asserts that

$$f(x) = u(x) + \mathbf{Vec} \int_0^1 t^2\, \overline{D}u(tx)\, x\, dt \tag{12}$$

is monogenic in B_4 such that $\mathbf{Sc}\, f = u$ in B_4. Hence

$$\begin{aligned} |f(x)| &\le |u(x)| + |x| \int_0^1 t^2 \left|\overline{D}u(tx)\right| \, dt \\ &\le |u(x)| + r \int_0^1 t^2 \left(\sum_{j=0}^{3} \left| \frac{\partial u}{\partial x_j}(tx) \right| \right) dt, \qquad x = r\zeta. \end{aligned}$$

By Minkowski's and the triangle inequalities

$$\begin{aligned} M_p(f;r) &\le M_p(u;r) + C \sum_{j=0}^{3} r \int_0^1 M_p\left(\frac{\partial u}{\partial x_j}; tr \right) dt \\ &\le M_p(u;r) + C \sum_{j=0}^{3} \int_0^r M_p\left(\frac{\partial u}{\partial x_j}; \rho \right) d\rho. \end{aligned}$$

Then by Lemmas 3 (i) and 5

$$\begin{aligned} \|f\|^p_{p,\alpha} &\le C \int_0^1 (1-r)^\alpha {M_p}^p(f;r)\, dr \\ &\le C\|u\|^p_{p,\alpha} + C \sum_{j=0}^{3} \int_0^1 (1-r)^\alpha \left(\int_0^r M_p\left(\frac{\partial u}{\partial x_j}; \rho \right) d\rho \right)^p dr \\ &\le C\|u\|^p_{p,\alpha} + C \sum_{j=0}^{3} \int_0^1 (1-r)^{\alpha+p} {M_p}^p\left(\frac{\partial u}{\partial x_j}; r \right) dr \\ &\le C\|u\|^p_{p,\alpha} + C \sum_{j=0}^{3} \left\| \frac{\partial u}{\partial x_j} \right\|^p_{p,\alpha+p} \le C\|u\|^p_{p,\alpha}. \end{aligned}$$

This completes the proof of Theorem 1. ∎

5. Maximal theorems in Bergman spaces on the unit ball of $\mathbb{R}^n$

From now on we focus our attention mainly on small indices $p \in (0,1)$ where we need some additional tools, in particular, a maximal theorem in Bergman spaces. We will need the following two auxiliary lemmas. The first of them is the well-known Whitney decomposition (see [27, Ch. 6]).

Lemma 6. *There exists a collection $\{\Delta_{kj}\}_{k,j}$, $1 \le j \le m_k$, $k = 0,1,2,\ldots,$ of closed cubes $\Delta_{kj} \subset B_n$ such that*

(i) $\bigcup_{k=0}^{\infty} \bigcup_{j=1}^{m_k} \Delta_{kj} = B_n$ *and* $\operatorname{diam} \Delta_{kj} \approx \operatorname{dist}(\Delta_{kj}, S)$.

(ii) *The interiors of all Δ_{kj} are pairwise disjoint. There exists another collection of extended cubes Δ^*_{kj} with the same center as Δ_{kj} such that the system $\{\Delta^*_{kj}\}_{k,j}$ forms a finite multiple covering of B_n. More precisely, each cube Δ^*_{kj} intersects at most 12^n cubes Δ_{kj}.*

Note that we can explicitly define Δ_{kj} as a "cube"

$$\Delta_{kj} = \left\{ x = r\zeta \in B_n \colon 1 - \frac{1}{2^k} \le r \le 1 - \frac{1}{2^{k+1}},\ \zeta \in S_{kj} \right\},$$

where S_{kj} is a part of the unit sphere chosen such that $\operatorname{diam} S_{kj} = c_n 2^{-k}$ with an absolute constant c_n depending only on n, and $\bigcup_{j=1}^{m_k} S_{kj} = S$ for each k. Furthermore, if y_{kj} ($y_{kj} = \rho_{kj}\xi_{kj}$) is the center of Δ_{kj}, then

$$|\Delta_{kj}| \approx |\Delta^*_{kj}| \approx (1 - |y_{kj}|)^n \approx 2^{-kn}.$$

Lemma 7. *Let Δ_{kj} and Δ^*_{kj} be some "cubes" from the previous lemma, and let $y_{kj} = \rho_{kj}\xi_{kj}$ be the center of Δ_{kj}. If a function u is harmonic in B_n, then for any $0 < p < \infty$ and $\alpha > -1$*

$$(1 - |y_{kj}|)^\alpha \max_{x \in \Delta_{kj}} |u(x)|^p \le \frac{C(p,\alpha,n)}{|\Delta^*_{kj}|} \int_{\Delta^*_{kj}} (1 - |y|)^\alpha |u(y)|^p \, dV_n(y).$$

Proof. Given a "cube" Δ_{kj} and any point $x \in \Delta_{kj}$ take the ball B_x with center x and radius 2^{-k-3} such that $B_x \subset \Delta^*_{kj}$. Then by the HL-property for the function $|u|^p$

$$|u(x)|^p \le \frac{C(p,n)}{|B_x|} \int_{B_x} |u(y)|^p \, dV_n(y), \qquad x \in \Delta_{kj}.$$

Since $|B_x| \approx |\Delta^*_{kj}|$,

$$\max_{x \in \Delta_{kj}} |u(x)|^p \le \frac{C(p,n)}{|\Delta^*_{kj}|} \int_{\Delta^*_{kj}} |u(y)|^p \, dV_n(y).$$

The result follows because $1 - |y_{kj}| \approx 1 - |y|$ for $y \in \Delta^*_{kj}$. ∎

We now need the weighted Bergman kernel K_α for the ball B_n [18, 19, 21, 22]:

$$K_\alpha(x,y) = \frac{2}{n\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1+k+n/2)}{\Gamma(k+n/2)} Z_k(x,y), \qquad x,y \in B_n, \tag{13}$$

where $\alpha > -1$, $Z_k(x,y)$ are extended zonal harmonics (see [3, Ch. 5 and 8]).

The following two maximal theorems in weighted Bergman spaces are of independent interest.

Theorem 2. *Let $u(x)\colon B_n \to \mathbb{R}$ be a real-valued harmonic function in the Bergman space h^p_α on the unit ball B_n for some $\alpha > -1$ and $0 < p < 1$. Then the radial type maximal function*

$$g(x) = \sup_{0<t<1} |\nabla u(y)|_{y=tx}| = \sup_{0<\rho<r} |(\nabla u)(\rho\zeta)|, \qquad x = r\zeta, \tag{14}$$

satisfies the inequality

$$\|g\|_{p+\alpha,\alpha} \le C(p,\alpha,n)\|u\|_{p,\alpha}.$$

Proof. By [21, p. 92], the continuous inclusion $h^p_\alpha \subset h^1_{(\alpha+n)/p-n}$ holds. For any $\beta \ge (\alpha+n)/p-n$, the function $u \in h^1_{(\alpha+n)/p-n}$ admits the integral representation (see [18, 19, 22])

$$u(x) = \int_{B_n} K_\beta(x,y)\, u(y)(1-|y|^2)^\beta \, dV_n(y), \qquad x \in B_n, \tag{15}$$

where K_β is the Bergman kernel (13). The gradient of the Bergman kernel can be estimated as follows ([22, Lem. 2.2], [21, Thm. 4.1], [18, Lem. 2.8])

$$|\nabla_x K_\beta(x,y)| \le \frac{C(\beta,n)}{|\rho x - \xi|^{\beta+n+1}}, \qquad x = r\zeta,\ y = \rho\xi.$$

Therefore, taking the gradient in (15), we obtain

$$|\nabla_x u(x)| \le C(\beta,n) \int_{B_n} |u(y)| \frac{(1-|y|)^\beta}{|\rho x - \xi|^{\beta+n+1}} \, dV_n(y), \qquad x \in B_n.$$

Now we can introduce the parameter $t \in (0,1)$ and use the simple inequality

$$|\xi - \rho x| \le 2|\xi - t\rho x|, \qquad x \in B_n,\ 0 < \rho, t < 1,\ \xi \in S,$$

which can be proved by the triangle inequality:

$$|\xi - \rho x| \le |\xi - t\rho x| + \rho|x| - t\rho|x| \le |\xi - t\rho x| + 1 - t\rho|x| \le 2|\xi - t\rho x|.$$

Hence

$$g(x) = \sup_{0<t<1} |\nabla u(tx)| \le C(\beta,n) \int_{B_n} |u(y)| \frac{(1-|y|)^\beta}{|\rho x - \xi|^{\beta+n+1}} \, dV_n(y), \qquad x \in B_n.$$

Now, we need a partition of the unit ball, the Whitney decomposition of B_n (see Lemma 6)

$$
\begin{aligned}
(16)\qquad g(x) &\le C(\beta,n)\sum_{k=0}^{\infty}\sum_{j=1}^{m_k}\int_{\Delta_{kj}}|u(y)|\frac{(1-|y|)^{\beta}}{|\rho x-\xi|^{\beta+n+1}}\,dV_n(y)\\
&\le C(\beta,n)\sum_{k=0}^{\infty}\sum_{j=1}^{m_k}(1-|y_{kj}|)^{\beta}|\Delta_{kj}|\sup_{y\in\Delta_{kj}}\frac{|u(y)|}{|\rho x-\xi|^{\beta+n+1}},
\end{aligned}
$$

where $y_{kj}=\rho_{kj}\xi_{kj}$ is the center of Δ_{kj}, and $y=\rho\xi$. Further, since we have $|y_{kj}|=1-3/2^{k+2}$,

$$
1-|x|+\frac{1}{2^{k+1}}|x|\le 1-|y_{kj}||x|\le 1-|x|+\frac{1}{2^{k}}|x|\le 2\left(1-|x|+\frac{1}{2^{k+1}}|x|\right).
$$

It follows that

$$
|\xi-\rho x|=|\zeta-ry|\approx|\zeta-ry_{kj}|=|\xi_{kj}-\rho_{kj}x|,\qquad x=r\zeta,\ y=\rho\xi.
$$

Raising both sides of (16) to the pth power,

$$
g^{p}(x)\le C(p,\beta,n)\sum_{k=0}^{\infty}\sum_{j=1}^{m_k}\frac{(1-|y_{kj}|)^{p\beta}|\Delta_{kj}|^{p}}{|\xi_{kj}-\rho_{kj}x|^{p(\beta+n+1)}}\sup_{y\in\Delta_{kj}}|u(y)|^{p},
$$

we can integrate and estimate it by Lemma 1 assuming β large enough (i.e. $\beta>(\alpha+n)/p-n$)

$$
\begin{aligned}
\|g\|_{p,\alpha+p}^{p} &\le C\sum_{k=0}^{\infty}\sum_{j=1}^{m_k}(1-|y_{kj}|)^{p\beta}|\Delta_{kj}|^{p}\sup_{y\in\Delta_{kj}}|u(y)|^{p}\int_{B_n}\frac{(1-|x|)^{\alpha+p}\,dV_n(x)}{\left|\xi_{kj}-\rho_{kj}x\right|^{p(\beta+n+1)}}\\
&\le C\sum_{k=0}^{\infty}\sum_{j=1}^{m_k}(1-|y_{kj}|)^{p\beta}|\Delta_{kj}|^{p}\frac{\sup_{y\in\Delta_{kj}}|u(y)|^{p}}{(1-\rho_{kj})^{p(\beta+n+1)-\alpha-p-n}}\\
&= C\sum_{k=0}^{\infty}\sum_{j=1}^{m_k}(1-|y_{kj}|)^{\alpha+n-pn}|\Delta_{kj}||\Delta_{kj}|^{p-1}\max_{y\in\Delta_{kj}}|u(y)|^{p}\\
&\le C\sum_{k=0}^{\infty}\sum_{j=1}^{m_k}(1-|y_{kj}|)^{\alpha}|\Delta_{kj}|\max_{y\in\Delta_{kj}}|u(y)|^{p}.
\end{aligned}
$$

By Lemma 7,

$$
\begin{aligned}
\|g\|_{p,\alpha+p}^{p} &\le C\sum_{k=0}^{\infty}\sum_{j=1}^{m_k}\int_{\Delta_{kj}^{*}}(1-|y|)^{\alpha}|u(y)|^{p}\,dV_n(y)\\
&\le C(p,\alpha,\beta,n)\int_{B_n}(1-|y|)^{\alpha}|u(y)|^{p}\,dV_n(y)\\
&= C(p,\alpha,\beta,n)\|u\|_{p,\alpha}^{p}.
\end{aligned}
$$

This completes the proof of Theorem 2. ■

Our second maximal theorem does not involve the gradient.

Theorem 3. *Let $u(x)\colon B_n \to \mathbb{R}$ be a real-valued harmonic function in the Bergman space h^p_α on the unit ball B_n for some $\alpha > -1$ and $0 < p < 1$. Then the radial type maximal function*

$$u_+(x) = \sup_{0<t<1} |u(tx)| = \sup_{0<\rho<r} |u(\rho\zeta)|, \qquad x = r\zeta,$$

satisfies the inequality

$$\|u_+\|_{p,\alpha} \le C(p,\alpha,n)\|u\|_{p,\alpha}.$$

Proof. Taking into account the integral formula (15) and using estimates of the Bergman kernel ([22, Lem. 2.2], [21, Thm. 4.1], [18, Lem. 2.7])

$$|K_\beta(x,y)| \le \frac{C(\beta,n)}{|\rho x - \xi|^{\beta+n}}, \qquad x = r\zeta,\ y = \rho\xi,$$

we then follow the lines of the proof of Theorem 2. Details will be omitted. ∎

Remark. Theorem 3 is recently proved in [12, Thm. 4] by a different method and in a more general setting. Our approach is based on Whitney decomposition and was applied in [1] in the context of upper half-space $\mathbb{R}^{n+1}_+$. Similar maximal theorems for harmonic Bergman spaces can also be found in [1, 2, 13, 20, 28, 29].

6. Harmonic conjugation, case $0 < p < 1$

Theorem 4. *Let $u(x) = u_0(x)\colon B_4 \to \mathbb{R}$ be a real-valued harmonic function in the unit ball B_4. If $u \in h^p_\alpha$ for some $\alpha > -1$ and $0 < p < 1$, then there exists a monogenic function $f(x)\colon B_4 \to \mathbb{H}$ such that $f \in \mathcal{M}^p_\alpha$ and $\mathbf{Sc}\, f = u$ in B_4, and*

$$\|f\|_{p,\alpha} \le C(p,\alpha)\|u\|_{p,\alpha}.$$

Proof. We are using again Sudbery's formula (12)

$$f(x) = u(x) + \mathbf{Vec}\int_0^1 t^2\, \overline{D}u(tx)\, x\, dt.$$

Since $|\overline{D}u| = |\nabla u|$,

$$\begin{aligned} |f(x)| &\le |u(x)| + |x| \int_0^1 t^2 \left|\overline{D}u(tx)\right| dt \\ &\le |u(x)| + r\int_0^1 |\nabla u(tx)|\, dt \\ &\le |u(x)| + 2r\int_0^1 g(tx)\, dt, \end{aligned}$$

where $g(x) = \sup_{0<t<1} |(\nabla u)(tx)|$ is the radial type maximal function (14). Because of the monotonicity of $g(r\zeta)$ in r, we may apply the inequality in Lemma 3 (ii) and obtain

$$\begin{aligned} |f(x)|^p &\leq |u(x)|^p + C_p r^p \int_0^1 (1-t)^{p-1} g^p(tx)\, dt \\ &= |u(x)|^p + C_p \int_0^r (r-t)^{p-1} g^p(t\zeta)\, dt, \end{aligned}$$

and

$$M_p{}^p(f;r) \leq M_p{}^p(u;r) + C_p \int_0^r (r-t)^{p-1} M_p{}^p(g;t)\, dt.$$

Consequently, by integrating and applying Fubini's Theorem and Lemma 4, we get

$$\begin{aligned} \|f\|_{p,a}^p &\leq \|u\|_{p,a}^p + C_p \int_0^1 (1-r)^\alpha \left(\int_0^r (r-t)^{p-1} M_p^p(g;t)\, dt \right) r^3\, dr \\ &= \|u\|_{p,a}^p + C_p \int_0^1 M_p{}^p(g;t) \left(\int_t^1 (r-t)^{p-1} (1-r)^\alpha r^3\, dr \right) dt \\ &\leq \|u\|_{p,a}^p + C(p,\alpha) \int_0^1 M_p{}^p(g;t)(1-t)^{\alpha+p}\, dt \\ &\leq \|u\|_{p,a}^p + C(p,\alpha) \|g\|_{p,a+p}^p. \end{aligned}$$

By Theorem 2 we have $\|g\|_{p,a+p} \leq C\|u\|_{p,a}$, which finishes the proof. ■

7. Complex-valued harmonic conjugates

In this section we identify the space $\mathbb{H}$ with the complex space $\mathbb{C}^2$ by means of the mapping that associates to $(z,w) = (x_0 + x_1\mathbf{i}, x_2 + x_3\mathbf{i})$ the quaternion $x = z + w\mathbf{j}$, where $z = x_0 + x_1\mathbf{i}$, $w = x_2 + x_3\mathbf{i}$. Observe that $z\mathbf{j} = \mathbf{j}\bar{z}$ for every $z \in \mathbb{C}$. A quaternion-valued function f can be written by means of its complex components as follows:

$$f(x) = f(z,w) = (u_0 + u_1\mathbf{i}) + (u_2 + u_3\mathbf{i})\mathbf{j} = U(x) + V(x)\mathbf{j},$$

where $U(x) = u_0(x) + u_1(x)\mathbf{i}$ and $V(x) = u_2(x) + u_3(x)\mathbf{i}$ are complex-valued functions of two complex variables z and w. On these complex-valued functions we consider related differential operators

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x_0} - \mathbf{i}\frac{\partial}{\partial x_1} \right), & \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial x_1} \right), \\ \frac{\partial}{\partial w} &= \frac{1}{2}\left(\frac{\partial}{\partial x_2} - \mathbf{i}\frac{\partial}{\partial x_3} \right), & \frac{\partial}{\partial \bar{w}} &= \frac{1}{2}\left(\frac{\partial}{\partial x_2} + \mathbf{i}\frac{\partial}{\partial x_3} \right). \end{aligned}$$

The Cauchy-Riemann-Fueter operator can then be written as follows:

$$\begin{aligned} Df &= \left(\frac{\partial f}{\partial x_0} + \mathbf{i}\frac{\partial f}{\partial x_1}\right) + \mathbf{j}\left(\frac{\partial f}{\partial x_2} - \mathbf{i}\frac{\partial f}{\partial x_3}\right) \\ &= \left[\left(\frac{\partial U}{\partial x_0} + \mathbf{i}\frac{\partial U}{\partial x_1}\right) + \left(\frac{\partial V}{\partial x_0} + \mathbf{i}\frac{\partial V}{\partial x_1}\right)\mathbf{j}\right] + \\ &\quad + \mathbf{j}\left[\left(\frac{\partial U}{\partial x_2} - \mathbf{i}\frac{\partial U}{\partial x_3}\right) + \left(\frac{\partial V}{\partial x_2} - \mathbf{i}\frac{\partial V}{\partial x_3}\right)\mathbf{j}\right] \\ &= 2\left(\frac{\partial U}{\partial \bar{z}} - \frac{\partial \bar{V}}{\partial \bar{w}}\right) + 2\left(\frac{\partial V}{\partial \bar{z}} + \frac{\partial \bar{U}}{\partial \bar{w}}\right)\mathbf{j}. \end{aligned}$$

Therefore the Cauchy-Riemann equations in the complex form can be written as

$$\frac{\partial \bar{V}}{\partial \bar{w}} = \frac{\partial U}{\partial \bar{z}} \qquad \text{and} \qquad \frac{\partial V}{\partial \bar{z}} = -\frac{\partial \bar{U}}{\partial \bar{w}},$$

or, equivalently,

$$\frac{\partial V}{\partial w} = \frac{\partial \bar{U}}{\partial z} \qquad \text{and} \qquad \frac{\partial V}{\partial \bar{z}} = -\frac{\partial \bar{U}}{\partial \bar{w}}. \tag{17}$$

Theorem 5. *Let $U\colon B_4 \to \mathbb{C}$ be a harmonic function in the unit ball B_4 and $U \in h^p_\alpha$ for some $\alpha > -1$ and $0 < p < \infty$. Then there exists a harmonic function $V\colon B_4 \to \mathbb{C}$ such that the function $f = U + V\mathbf{j}$ belongs to the monogenic Bergman space $\mathcal{M}^p_\alpha$, and*

$$\|f\|_{p,\alpha} \le C(p,\alpha)\|U\|_{p,\alpha}.$$

Proof. Given a complex-valued harmonic function U, the compatibility condition

$$\frac{\partial^2 \bar{U}}{\partial z \partial \bar{z}} = -\frac{\partial^2 \bar{U}}{\partial w \partial \bar{w}}$$

for the system (17) is satisfied. Hence, there exists a solution V (see, e.g. [24, Ch. 16]) harmonic in B_4 ,

$$\frac{1}{4}\Delta V = \frac{\partial^2 V}{\partial z \partial \bar{z}} + \frac{\partial^2 V}{\partial w \partial \bar{w}} = -\frac{\partial^2 \bar{U}}{\partial z \partial \bar{w}} + \frac{\partial^2 \bar{U}}{\partial z \partial \bar{w}} = 0.$$

So, the function $f(x) = U(x) + V(x)\mathbf{j} = u_0 + u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is monogenic in B_4. It is clear that since $U = u_0 + u_1\mathbf{i}$ is in h^p_α, then so does for u_0 and u_1 and $\|U\|_{p,\alpha} \approx \|u_0\|_{p,\alpha} + \|u_1\|_{p,\alpha}$. Therefore, we obtain by Theorems 1 and 4,

$$\|f\|_{p,\alpha} \le C\|u_0\|_{p,\alpha} \le C\|U\|_{p,\alpha}.$$

This completes the proof of Theorem 5. ∎

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