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On uniform approximation in real non-degenerate Weil polyhedron

Albert I. Petrosyan

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Summary: It is proved that any function holomorphic in a real, non-degenerate Weil polyhedron G and continuous in its closure $\overline{G}$ can be uniformly approximated by functions holomorphic in a neighborhood of $\overline{G}$. Besides, it is proved that such functions can be approximated by polynomials if G is a polynomial polyhedron.

1 Introduction

1. This paper is studies the possibility of uniform approximation of functions holomorphic in a Weil polyhedron $G \subset \mathbb{C}^n$ and continuous in its closure $\overline{G}$. It is proved that if G is real non-degenerate, then any function holomorphic in G and continuous in $\overline{G}$ can be uniformly approximated by functions holomorphic in a neighborhood of $\overline{G}$ (Theorem 3.1). Besides, it is proved that such functions can be approximated by polynomials if G is a polynomial polyhedron (Theorem 3.2). Note that the class of real non-degenerate polyhedrons is wide enough to provide approximation of any holomorphity domain by real non-degenerate polyhedrons.

The following method is used: after establishing the possibility of local approximation (Lemma 2.2), a global approximating function is "glued" from functions realizing local approximations (Theorem 3.1). To this end, an improvement of an argument from [1] is used for strictly pseudoconvex domains. This argument is based on some uniform estimates [2], [3] for the solutions of the equation $\overline{\partial} u = g$, where g is a $\overline{\partial}$-closed differential form of $(0, 1)$-type in G. Note that for arbitrary Weil polyhedrons there is no theorem on approximation. Besides, using a different method the author [4] has proved an approximation theorem under the requirement on complex non-degeneracy.

It turns out that if $n = 2$, then for the validity of the main result of paper it suffices to require that the non-degeneracy condition is satisfied only on the distinguished boundary of G. Nevertheless, in this case (Theorem 4.1) G is required to be a special polyhedron, i.e. the number of functions determining G equals 2 (to the dimension of $\mathbb{C}^2$).

2. A bounded domain $G \subset \mathbb{C}^n$ is called Weil polyhedron if there are some functions $\chi_1, \ldots, \chi_N$ holomorphic in a neighborhood V of $\overline{G}$ and such that

$$G = \{z \in V\colon\ |\chi_i(z)| < 1, \quad i = 1, 2, \ldots, N, \quad N \geq n\}. \tag{1.1}$$

The boundary ∂G of G consists of the $(2n-1)$-dimensional “edges"

$$\sigma_i = \{z \in \partial G: \ |\chi_i(\zeta)| = 1\}$$

intersecting along the k-dimensional “ribs"

$$\sigma_{i_1,\dots,i_k} = \sigma_{i_1} \cap \cdots \cap \sigma_{i_k}.$$

The union of all the n-dimensional ribs is the distinguished boundary of G. The domain G is called polynomial polyhedron if all determining functions χ_i are polynomials in (1.1).

Definition 1.1 *We call a polyhedron* (1.1) *real non-degenerate if for any collection* $i_1, \dots, i_k$ *the matrix*

$$\left(\mathrm{grad}_{\mathbf{R}} |\chi_{i_1}(z)|, \dots, \mathrm{grad}_{\mathbf{R}} |\chi_{i_k}(z)|\right)$$

attains its maximal rank in all points $z \in \sigma_{i_1,\dots,i_k}$. *Here*

$$\mathrm{grad}_{\mathbf{R}}\, \chi(z) = {}^t\left(D_1\chi(z), \dots, D_n\chi(z), \overline{D}_1\chi(z), \dots, \overline{D}_n\chi(z)\right),$$

where the superscript t before the bracket means transposition and

$$D_k\chi(z) = \frac{\partial\chi(z)}{\partial z_k}, \quad \overline{D}_k\chi(z) = \frac{\partial\chi(z)}{\partial \overline{z}_k}, \quad k = 1, \dots, n.$$

Geometrically, Definition 1.1 means that the edges $\sigma_{i_1}, \dots, \sigma_{i_k}$ intersect in a general position (in the real analysis sense).

2 Local approximation

We start by proving the following geometrical property of non-degenerate polyhedrons.

Proposition 2.1 *Let G be a real non-degenerate polyhedron* (1.1) *and let $N \le 2n$. Then for any point $\zeta \in \partial G$ there exist a neighborhood B_ζ and a vector ν_ζ such that $z + \delta\nu_\zeta \in G$ if $z \in \overline{B}_\zeta \cap \overline{G}$ for sufficiently small $\delta > 0$.*

Proof: Denote $\varphi_i = |\chi_i| - 1$ and assume that the point $\zeta \in \partial G$ belongs to the edge $\sigma_{i_1,\dots,i_k}$, i.e. $\varphi_{i_1}(\zeta) = 0, \dots, \varphi_{i_k}(\zeta) = 0$ and

$$\varphi_s(\zeta) < 0, \quad s \neq i_1, \dots, i_k. \tag{2.1}$$

By $k \le 2n$ and our assumptions, the vectors $\mathrm{grad}_{\mathbf{R}}\, \varphi_{i_1}(\zeta), \dots, \mathrm{grad}_{\mathbf{R}}\, \varphi_{i_k}(\zeta)$ are linearly independent. Hence, there is some point w such that

$$\sum_{m=1}^{n} D_m\varphi_j(\zeta)(w_m - \zeta_m) + \sum_{m=1}^{n} \overline{D}_m\varphi_j(\zeta)\left(\overline{w}_m - \overline{\zeta}_m\right) < 0, \quad j = i_1, \dots, i_k.$$

Due to the continuity of $D_m\varphi_j(\zeta)$ and $\overline{D}_m\varphi_j(\zeta)$, there is a neighborhood B_ζ, such that

$$\sum_{m=1}^{n} D_m\varphi_j(z)(w_m-\zeta_m)+\sum_{m=1}^{n}\overline{D}_m\varphi_j(z)\left(\overline{w}_m-\overline{\zeta}_m\right)<0,\quad j=i_1,\dots,i_k \tag{2.2}$$

for all points $z\in\overline{B}_\zeta$. If $z\in\overline{B}_\zeta$, $\delta>0$, then

$$\varphi_j(z+\delta(w-\zeta))=\varphi_j(z)+2\delta\mathrm{Re}\sum_{m=1}^{n}D_m\varphi_j(z)(w_m-\zeta_m)+o(\delta). \tag{2.3}$$

Denoting $\nu_\zeta=w-\zeta$ and taking into account that $\varphi_j(z)\le 0$ for $z\in\overline{G}$, from (2.2) and (2.3) we conclude that there exists some $\delta_0>0$ such that for $\delta<\delta_0$

$$\varphi_j\left(z+\delta\nu_\zeta\right)<0,\quad j=i_1,\dots,i_k,\quad z\in\overline{B}_\zeta\cap\overline{G}. \tag{2.4}$$

By continuity of φ_j, it follows from (2.1) that one can choose a neighborhood B_ζ and a number δ_0 such that for $\delta<\delta_0$

$$\varphi_s\left(z+\delta\nu_\zeta\right)<0,\quad s\ne i_1,\dots,i_k,\quad z\in\overline{B}_\zeta\cap\overline{G}.$$

Hence, by (2.4) we conclude that $z+\delta\nu_\zeta\in G$. □

Let $A(G)$ be the uniform algebra of functions holomorphic in G and continuous in $\overline{G}$. Recalling that a function is said to be holomorphic in a compact set K if it is holomorphic in some neighborhood of K, we prove

Lemma 2.2 *There exists a finite covering* $\{U_k\colon k=0,1,\dots,p\}$ *of* $\overline{G}$ *by open sets, such that for any* $\varepsilon>0$ *and any* $f\in A(G)$ *there are some functions* f_k *holomorphic in* $\overline{U_k}\cap\overline{G}$ *and such that*

$$\sup_{z\in\overline{U}_k\cap\overline{G}}|f(z)-f_k(z)|<\varepsilon. \tag{2.5}$$

Proof: Let $f\in A(G)$, $\zeta\in\partial G$ and let B_ζ be a neighborhood satisfying the conditions of Proposition 2.1. Then the family of open sets $\{B_\zeta:\ \zeta\in\partial G\}$ covers the compact ∂G, and a finite subcovering $\{B_{\zeta_k},\ k=1,\dots,p\}$ can be chosen. By Proposition 2.1, the functions $f(z+\delta\nu_{\zeta_k})$ are holomorphic in $\overline{B}_{\zeta_k}\cap\overline{G}$ for any sufficiently small $\delta>0$. Besides, by the uniform continuity of f in $\overline{G}$

$$\sup_{z\in\overline{B}_{\zeta_k}\cap\overline{G}}\left|f\left(z+\delta\nu_{\zeta_k}\right)-f(z)\right|\to 0\quad\text{as}\quad\delta\to 0.$$

Now, choosing $\delta>0$ small enough and denoting $U_k=B_{\zeta_k}$, $f_k(z)=f(z+\delta\nu_{\zeta_k})$, we get (2.5) for $k=1,\dots,p$. Further, taking a compact subdomain $U_0\subset G$ such that the system $\{U_k\colon k=0,1,\dots,p\}$ is an open covering of $\overline{G}$ and setting $f_0(z)=f(z)$ we conclude that (2.5) obviously is true also for $k=0$. □

3 Global approximation

The following Theorem is the main result.

Theorem 3.1 *Let G be a real non-degenerate Weil polyhedron* (1.1) *and let $N \leq 2n$. Then any function $f \in A(G)$ can be uniformly approximated in $\overline{G}$ by functions holomorphic in $\overline{G}$.*

Proof: Let $\varepsilon > 0$, let $f \in A(G)$ and let $\{U_k : k = 0, 1, \ldots, p\}$ be those of Lemma 2.2. Then by Lemma 2.2, there are functions f_k holomorphic in $\overline{U}_k \cap \overline{G}$ and such that

$$\| f_k - f \|_{U_k \cap G} < \varepsilon, \quad k = 0, 1, \ldots, p. \tag{3.1}$$

Now suppose $\{e_k(z),\ k = 0, 1, \ldots, p\}$ is a partition of unity, i.e. a system of infinitely differentiable, nonnegative, finite functions such that

(a) Supp $e_k \subset U_k, \quad k = 0, 1, \ldots, p$,

(b) $\sum_{k=0}^{p} g_k(z) \equiv 1$ in some neighborhood of $\overline{G}$.

Then, choose $\eta(\varepsilon) > 0$ small enough to provide the holomorphity of f_k in the sets

$$V_k = U_k \cap G^{\varepsilon}, \quad k = 0, 1, \ldots, p,$$

where

$$G^{\varepsilon} = \{z \in V : \ |\chi_i(z)| < 1 + \eta(\varepsilon), \quad i = 1, 2, \ldots, N\}.$$

It is obvious that

$$\| f_k - f_i \|_{U_k \cap U_i \cap G} \leq \| f_k - f \|_{U_k \cap G} + \| f_i - f \|_{U_i \cap G} < 2\varepsilon, \ i, k = 0, 1, \ldots, p, \tag{3.2}$$

and, if necessary, taking smaller $\eta(\varepsilon) > 0$ we can get

$$\| f_k - f_i \|_{V_k \cap V_i} < 3\varepsilon, \quad k, i = 0, 1, \ldots, p, \tag{3.3}$$

by continuity. Now consider the functions

$$h_{ik}(z) = \begin{cases} [f_i(z) - f_k(z)]e_k(z) & \text{if } z \in V_i \cap V_k; \\ 0 & \text{if } z \in V_i \setminus V_k, \end{cases}$$

$$h_i(z) = \sum_{k=0}^{p} h_{ik}(z). \tag{3.4}$$

One can see that the support of $g_k(z)$ belongs to the set B_k (by the assumption (a)) and the set $V_i^{\varepsilon} \bigcap \partial V_k^{\varepsilon}$ does not intersect with this support. Therefore, the functions h_{ik}^{ε} and h_i^{ε} are infinitely differentiable in V_i^{ε}, and by (3.3)

$$|h_i(z)| \leq \sum_{k=0}^{p} | f_i(z) - f_k(z)| e_k(z) < 3\varepsilon \sum_{k=0}^{p} g_k(z) = 3\varepsilon \tag{3.5}$$

for all $z \in V_i^\varepsilon \bigcap \overline{G^\varepsilon}$. Further, if $z \in V_i \cap V_j$, then

$$\begin{aligned} h_i(z) - h_j(z) &= \sum_{k=0}^{p} \big[f_i(z) - f_k(z) \big] e_k(z) - \sum_{k=0}^{p} \big[f_j(z) - f_k(z) \big] e_k(z) \\ &= \sum_{k=0}^{p} \big[f_i(z) - f_j(z) \big] e_k(z) = f_i(z) - f_j(z), \end{aligned}$$

i.e.

$$f_i(z) - h_i(z) = f_j(z) - h_j(z), \quad i, j = 0, 1, \ldots, p.$$

This means that the function

$$\psi(z) = f_i(z) - h_i(z) \quad \text{if} \quad z \in V_i, \tag{3.6}$$

is globally given in G^ε, and moreover $h \in C^\infty(G^\varepsilon)$. Besides, using the inequalities (3.5) and (3.1), from (3.6) we obtain

$$|\psi(z) - f(z)| \le |h_i(z)| + |f_i(z) - f(z)| < 4\varepsilon, \quad z \in U_i \cap \overline{G}.$$

Consequently,

$$\|\psi - f\|_G < 4\varepsilon. \tag{3.7}$$

Considering the differential form $g = \overline{\partial}\psi$ in the domain G^ε, one can see that $\overline{\partial} g = 0$. Besides, by (3.4) and the holomorphity of f_i in V_i

$$g = \overline{\partial}\psi(z) = \overline{\partial} h_i(z) = \sum_{k=0}^{p} \overline{\partial} h_{ik}(z) = \sum_{k=0}^{p} (f_i(z) - f_k(z))\, \overline{\partial} e_k(z) \tag{3.8}$$

for $z \in V_i \cap \overline{G^\varepsilon}$. In addition, if it is set $\gamma_0 = \gamma_0(G) = \max\limits_{0 \le k \le p} \|\overline{\partial} e_k\|_{U_k}$, then by (3.8) and (3.3)

$$\|g\|_{G^\varepsilon} \le \sum_{k=0}^{p} \|f_i - f_k\|_{G^\varepsilon} \left\|\overline{\partial} e_k\right\|_{U_k} \le 3\gamma_0\varepsilon. \tag{3.9}$$

Now consider the equation $\overline{\partial} u = g$. In [2, 3] it is proved that in the domain G^ε there exists a solution u_0 of this equation, which permits the uniform estimate

$$\|u_0\|_{G^\varepsilon} \le \gamma(G^\varepsilon)\|g\|_{G^\varepsilon}. \tag{3.10}$$

One can be convinced that from the proof of (3.10) it follows that the constants $\gamma(G^\varepsilon)$ are bounded, i.e.

$$\gamma(G^\varepsilon) \le \gamma = \gamma(G). \tag{3.11}$$

Besides, (3.10), (3.9) and (3.11) imply

$$\|u_0\|_{G^\varepsilon} \le 3\gamma_0\gamma\varepsilon. \tag{3.12}$$

Further, the function $F(z) = \psi(z) - u_0(z)$ is holomorphic in the domain G^ε since $\overline{\partial}\psi - \overline{\partial}u_0 = g - \overline{\partial}u_0 = 0$. Besides, by (3.7) and (3.12)

$$\| f - F\|_G \le \|\psi - f\|_G + \|u_0\|_G < 4\varepsilon + 3\gamma_0\gamma\varepsilon = \gamma_1\varepsilon,$$

where the constant γ_1 depends solely on G. □

For polynomial polyhedrons a stronger assertion than Theorem 3.1 is true. Before proving this assertion, recall that a compact set K is said to be polynomially convex if for any point $\zeta \notin K$ there is a polynomial P_ζ such that $|P_\zeta(\zeta)| > \max_{z\in K} |P_\zeta(z)|$. Besides, the Oka–Weil approximation theorem (see, e.g. [5]), states that *any function holomorphic in a neighborhood of a polynomially convex compact set K can be uniformly approximated on K by polynomials.*

Theorem 3.2 *Let G be a real non-degenerate polynomial polyhedron* (1.1) *and let $N \le 2n$. Then any function $f \in A(G)$ can be uniformly approximated on $\overline{G}$ by polynomials.*

Proof: Suppose $\zeta \notin \overline{G}$ and note that $|\chi_i(\zeta)| > 1$ for some i by the definition of the polyhedron G, i.e. $\overline{G}$ is polynomially convex compact set. It remains to see that the desired assertion follows from Theorem 3.1 and the Oka–Weil theorem. □

4 Non-degenerate distinguished boundary

Below we shall assume that $D \subset \mathbb{C}^2$ is a special polyhedron

$$D = \left\{z \in \mathbb{C}^2 :\ |\chi_i(z)| < 1,\ i = 1, 2\right\} \tag{4.1}$$

with real non-degenerate distinguished boundary $\Gamma = \{z \in \partial D:\ |\chi_1(z)| = |\chi_2(z)| = 1\}$, i.e.

$$\operatorname{rank}(\operatorname{grad}_{\mathbf{R}} |\chi_1(z)|,\ \operatorname{grad}_{\mathbf{R}} |\chi_2(z)|) = 2, \ \text{ if } z \in \Gamma. \tag{4.2}$$

The following Theorem relates to approximation on special Weil polyhedrons with real non-degenerate distinguished boundary. Note, that the set of special polyhedrons is wide enough: due to Bishop's theorem (see [6]) any polyhedron can be approximated by special ones.

Theorem 4.1 *Let the polyhedron* (4.1) *satisfy* (4.2)*. Then any function $f \in A(D)$ can be uniformly approximated in $\overline{D}$ by functions holomorphic in $\overline{G}$. If D is a polynomial polyhedron any function $f \in A(D)$ can be approximated by polynomials.*

Proof: By continuity, it is possible to choose a number $\delta > 0$ small enough to provide that the polyhedron

$$G = \left\{z \in \mathbb{C}^2 :\ 1 - \delta < |\chi_i(z)| < 1,\ i = 1, 2\right\}$$

(in a sense “attached" to Γ) be real nondegenerating. One can be convinced that Theorem 3.1 is true for G. Hence for any $f \in A(D) \subset A(G)$ and any $\varepsilon > 0$ there is a function F_ε holomorphic in some neighborhood of $\overline{G}$ and such that

$$\max_{z\in\overline{G}} |f(z) - F_\varepsilon(z)| < \varepsilon. \tag{4.3}$$

Further, choose a number $\eta = \eta(\varepsilon) > 0$ such that closure of the polyhedron

$$G_\varepsilon = \left\{z \in \mathbb{C}^2 :\ 1 - \delta < |\chi_i(z)| < 1 + \eta(\varepsilon),\ i = 1, 2\right\}$$

belongs to the mentioned neighborhood. Then F_ε is holomorphic in $\overline{G}_\varepsilon$. Further, introducing the notations

$$\begin{aligned}
\gamma_1 &= \left\{z \in \overline{G}_\varepsilon :\ |\chi_1(z)| = |\chi_2(z)| = 1 + \eta(\varepsilon)\right\};\\
\gamma_2 &= \left\{z \in \overline{G}_\varepsilon :\ |\chi_1(z)| = |\chi_2(z)| = 1 - \delta\right\};\\
\gamma_3 &= \left\{z \in \overline{G}_\varepsilon :\ |\chi_1(z)| = 1 - \delta,\ |\chi_2(z)| = 1 + \eta(\varepsilon)\right\};\\
\gamma_4 &= \left\{z \in \overline{G}_\varepsilon :\ |\chi_1(z)| = 1 + \eta(\varepsilon),\ |\chi_2(z)| = 1 - \delta\right\},
\end{aligned}$$

observe that the distinguished boundary of the polyhedron G_ε is the union of $\gamma_1 - \gamma_4$ oriented in an appropriate way. Therefore, according to the Bergman–Weil integral representation (see [7, 8]), for a $z \in G_\varepsilon$ we have

$$F_\varepsilon(z) = \sum_{k=1}^{4} \frac{1}{4\pi_2} \int_{\gamma_k} F_\varepsilon(\zeta) \frac{J(\zeta, z)d\zeta_1 \wedge d\zeta_2}{[\chi_1(\zeta) - \chi_1(z)][\chi_2(\zeta) - \chi_2(z)]} = \sum_{k=1}^{4} I_k(z), \tag{4.4}$$

where

$$J(\zeta, z) = \begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix},$$

and P_{k1} and P_{k2} are Hefer’s coefficients of the polynomial χ_k, which are defined by the decomposition

$$\chi_k(\zeta) - \chi_k(z) = (\zeta_1 - z_1)P_{k1}(\zeta, z) + (\zeta_2 - z_2)P_{k2}(\zeta, z), \quad k = 1, 2.$$

One can see that the summand $I_1(z)$ in the right-hand side of (4.4) is holomorphic in a neighborhood of $\overline{D}$ since for $\zeta \in \gamma_1$ and $z \in \overline{D}$ the denominator

$$[\chi_1(\zeta) - \chi_1(z)][\chi_2(\zeta) - \chi_2(z)]$$

of the integrand of $I_1(z)$ does not vanish. Besides from the below proposition it follows that

$$\max_{z\in\Gamma} |F_\varepsilon(z) - I_1(z)| \le \sum_{k=1}^{3} \max_{z\in\Gamma} |I_k(z)| < c_1\varepsilon, \tag{4.5}$$

where the constant c_1 is independent of ε. Hence, by the maximum principle and (4.3)

$$\max_{z\in\overline{D}} | f(z) - I_1(z)| = \max_{x\in\Gamma} | f(z) - I_1(z)| < (1 + c_1)\varepsilon,$$

i.e. in $\overline{D}$ the function f is uniformly approximated by functions holomorphic in $\overline{D}$. For polynomial polyhedrons our statement is proved as that in Theorem 3.2. □

Proposition 4.2 *The following estimates are true for the summands of* (4.4)*:*

$$\max_{z\in\Gamma} |I_k(z)| < c\varepsilon, \quad k = 2, 3, 4,$$

where c is a constant independent of ε.

Proof: Obviously γ_2 is the distinguished boundary of the polyhedron

$$\{z \in \mathbb{C}^2 : \ |\chi_i(z)| < 1 - \delta, \ i = 1, 2\}.$$

Therefore, by our assumptions the function f is holomorphic in the closure of γ_2, and according to of Bergman–Weil formula

$$\frac{1}{4\pi^2} \int_{\gamma_2} f(\zeta) \frac{J(\zeta, z) d\zeta_1 \wedge d\zeta_2}{[\chi_1(\zeta) - \chi_1(z)][\chi_2(\zeta) - \chi_2(z)]} = 0 \quad \text{for} \quad z \in \Gamma.$$

Thus

$$\begin{aligned} I_2(z) &= \frac{1}{4\pi^2} \int_{\gamma_2} F_\varepsilon(\zeta) \frac{J(\zeta, z) d\zeta_1 \wedge d\zeta_2}{[\chi_1(\zeta) - \chi_1(z)][\chi_2(\zeta) - \chi_2(z)]} \\ &= \frac{1}{4\pi^2} \int_{\gamma_2} [F_\varepsilon(\zeta) - f(\zeta)] \frac{J(\zeta, z) d\zeta_1 \wedge d\zeta_2}{[\chi_1(\zeta) - \chi_1(z)][\chi_2(\zeta) - \chi_2(z)]}. \end{aligned}$$

As $\gamma_2 \subset \overline{G}$, from (4.3) we obtain

$$|I_2(z)| \le \frac{\varepsilon}{4\pi_2} \int_{\gamma_2} \left| \frac{J(\zeta, z) d\zeta_1 \wedge d\zeta_2}{[\chi_1(\zeta) - \chi_1(z)][\chi_2(\zeta) - \chi_2(z)]} \right|, \quad z \in \Gamma. \tag{4.6}$$

Further, it is obvious that

$$|\chi_k(\zeta) - \chi_k(z)| \ge \big| |\chi_k(\zeta)| - |\chi_k(z)| \big| = \delta, \quad k = 1, 2,$$

for $z \in \Gamma$ and $\zeta \in \gamma_2$. Therefore, by (4.6)

$$\max_{z\in\Gamma} |I_2(z)| < c\varepsilon. \tag{4.7}$$

To estimate the integrals I_3 and I_4, consider the following real, three-dimensional edge of the polyhedron G_ε:

$$\sigma = \{\zeta \in G_\varepsilon : \ |\chi_1(\zeta)| = 1 - \delta, \quad 1 - \delta < |\chi_2(\zeta)| < 1 + \eta(\varepsilon)\}.$$

This edge is bounded by the cycles γ_2 and γ_3. Besides, for a fixed $z_0 \in \Gamma$ the set of singularities of the differential form

$$\omega = F_\varepsilon(\zeta) \frac{J(\zeta, z_0) d\zeta_1 \wedge d\zeta_2}{[\chi_1(\zeta) - \chi_1(z_0)][\chi_2(\zeta) - \chi_2(z_0)]},$$

which lie on σ, is the curve

$$P_{z_0} = \{\zeta \in \sigma : \; |\chi_1(\zeta)| = 1 - \delta_1, \quad \chi_2(\zeta) = \chi_2(z_0)\},$$

and the mentioned form is closed in $\sigma \setminus P_{z_0}$. Besides, by Stocks' formula

$$\int_{\gamma_2} \omega - \int_{\gamma_3} \omega = \int_{\sigma} d\omega + \int_{P_{z_0}} \text{res } \omega,$$

or, which is the same,

$$I_2(z_0) - I_3(z_0) = \int_{P_{z_0}} \text{res } \omega.$$

Hence, by (4.7) the desired estimate of I_3 follows from the estimate of $\int_{P_{z_0}} \text{res } \omega$. Besides, one can be convinced that

$$\text{res } \omega = \frac{F_\varepsilon(\zeta) J(\zeta, z_0)}{\chi_1(\zeta) - \chi_1(z_0)} \frac{d\zeta_1}{D_2 \chi_2(\zeta)} = -\frac{F_\varepsilon(\zeta) J(\zeta, z_0)}{\chi_1(\zeta) - \chi_1(z_0)} \frac{d\zeta_2}{D_1 \chi_2(\zeta)}, \tag{4.8}$$

where the second equality holds since $\chi_2(\zeta) \equiv \chi_2(z_0)$ for $\zeta \in P_{z_0}$ and hence $d\chi_2 = 0$ in P_{z_0}, i.e. $\dfrac{d\zeta_1}{D_2\chi_2(\zeta)} = \dfrac{d\zeta_2}{D_1\chi_2(\zeta)}$. Replacing F_ε by f in (4.8), we come to the differential form

$$\Omega = \frac{f(\zeta) J(\zeta, z_0)}{\chi_1(\zeta) - \chi_1(z_0)} \frac{d\zeta_1}{D_2 \chi_2(\zeta)} = -\frac{f(\zeta) J(\zeta, z_0)}{\chi_1(\zeta) - \chi_1(z_0)} \frac{d\zeta_2}{D_1 \chi_2(\zeta)}$$

which is holomorphic in all points of the analytic subvariety

$$T_{z_0} = \{\zeta : \; |\chi_1(\zeta)| < 1 - \delta \;\; \chi_2(\zeta) = \chi_2(z_0)\},$$

except the singularities $M \cap T_{z_0}$, where

$$M = \left\{\zeta \in \mathbb{C}^2 : \;\; \text{grad } \chi_2(\zeta) = 0\right\}.$$

Observe that each component of the set M is either a point or a level surface of the function χ_2 (obviously differing from the level $\chi_2(\zeta) = \chi_2(z_0)$). Thus, T_{z_0} can contain only single-point components of the set M, and hence for all z from a certain neighborhood of a point z_0 (except z_0), lying on the surface T_z, the form Ω has no singularities.

Now note that $\partial T_z = P_z$ and

$$\int_{P_z} \Omega = \int_{T_z} d\Omega = 0$$

by Stocks' formula. It is obvious that the integral $\int_{P_z} \Omega$ continuously depends on z (at least in some neighborhood of z_0), and hence $\int_{P_{z_0}} \Omega = 0$. Therefore, taking into account that $|\chi_1(\zeta) - \chi_1(z_0)| \geq \delta$ for $\zeta \in P_{z_0}$ we obtain

$$\left| \int_{P_{z_0}} \text{res } \omega \right| = \left| \int_{P_{z_0}} \text{res } \omega - \int_{P_{z_0}} \Omega \right| = \left| \int_{P_{z_0}} \frac{[F_\varepsilon(\zeta) - f(\zeta)]}{[\chi_1(\zeta) - \chi_1(z_0)]} \frac{J(\zeta, z_0)\, d\zeta_1}{D_2 \chi_2(\zeta)} \right| < c\varepsilon.$$

Now it remains to observe that if in the argument above the derivative $D_2\chi_2$ vanishes in some points of P_{z_0}, then $D_1\chi_2 \neq 0$ in these points, and we replace $\frac{d\zeta_1}{D_2\chi_2(\zeta)}$ by $-\frac{d\zeta_2}{D_1\chi_2(\zeta)}$. The integral $I_4(z)$ is estimated similarly. □

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Albert Petrosyan
Faculty of Mathematics
Yerevan State University
1 Aleck Manoogian street
375049 Yerevan
Armenia
albpet@xter.net