On A^p_ω spaces in the unit ball of $\mathbb{C}^n$

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Abstract

This paper relates to arbitrarily large A^p_ω spaces in the unit ball of $\mathbb{C}^n$. The introduced M.M.Djrbashian multidimensional kernel and the used technics allow to obtain the similarities of the representations proved in the early works of M.M.Djrbashian [1], [2] (1945–1948), which in essence gave rise to the theory of A^p_α (or initially $H^p(\alpha)$) spaces in the unit disc of the complex plane, and hence to extend to $\mathbb{C}^n$ several results of the one-dimensional general theory of [3] (see also [4]). Nonetheless, the paper gives only the representation connected with a natural isometry between A^2_ω and the ordinary Hardy space H^2 in the ball, which has an explicit form of integral operator along with its inversion.

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1 The spaces A^p_ω

Everywhere below $B = \{z \in \mathbb{C}^n : |z| < 1\}$ is the open unit ball in $\mathbb{C}^n$ and S is the unit sphere in $\mathbb{C}^n$; ν is the normalized Lebesgue measure in $\mathbb{C}^n$, so that $\nu(B) = 1$; σ is the normalized surface-area measure on S, so that $\sigma(S) = 1$. For any $z \in \mathbb{C}^n$ and any multi-index $s = (s_1, \dots, s_n)$, let $z^s = z_1^{s_1} z_2^{s_2} \dots z_n^{s_n}$, $s! = s_1! s_2! \dots s_n!$, $|s| = s_1 + s_2 + \dots + s_n$. Further, by Ω denotes the class of functions $\omega(t)$ in $[0,1]$ such that $\omega(1) = \omega(1-0)$ and

(i) $0 < \bigvee_\delta^1 \omega < \infty$ for any $\delta \in [0,1)$;

(ii) $\Delta_m \equiv \Delta_m(\omega) = -\int_0^1 t^m d\omega(t) \neq 0, \infty, \quad m = 0, 1, \dots;$

(iii) $\liminf_{m\to\infty} \sqrt[m]{|\Delta_m|} \geq 1$.

For $\omega \in \Omega$ consider the function

$$C_\omega(z,w) = \sum_s \frac{z^s \overline{w}^s}{\gamma_s \Delta_{|s|}}, \quad \text{where } \gamma_s = \frac{(n-1)!s!}{(n-1+|s|)!}. \tag{1}$$

Using Stirling's formula and (i), (iii), one can see that for any fixed $w \in \overline{B}$ the multiple power series in (1) converges uniformly on compact subsets of B. Hence $C_\omega(\cdot, w)$ is holomorphic in B.

For a given $\omega \in \Omega$ denote $d\mu_\omega(w) = -d\omega(r^2)\, d\sigma(\zeta)$, where $w = r\zeta$ is the polar form of $w \in B$, (i.e. $r = |w|$, $\zeta \in S$), and define $L^p_\omega = L^p_\omega(B)$ as the set of all measurable by $d\mu_\omega$ functions in B, for which

$$\|f\|_{p,\omega} = \left\{ \int_B |f(w)|^p \, |d\mu_\omega(w)| \right\}^{1/p} < +\infty, \quad 0 < p < \infty. \tag{2}$$

It is well known that L^p_ω is a Banach space with the norm $\|f\|_{p,\omega}$ if $1 \le p < \infty$ and L^p_ω is a complete metric space with $\rho(f,g) = \|f - g\|^p_{p,\omega}$ if $0 < p < 1$. Denoting the holomorphic subset of L^p_ω by $A^p_\omega = A^p_\omega(B)$, we start by

Proposition 1. *Let $0 < p < \infty$ and let K be a compact subset of B. Then there exist a constant $C \equiv C(K, p, \omega)$ such that*

$$\max_{z \in K} |f(z)| \le C \|f\|_{p,\omega}, \quad f \in A^p_\omega. \tag{3}$$

Proof. Choose $r \in (0,1)$ such that $K \subset rB$ and let $R \in (r, 1)$. Then obviously

$$\left| \frac{R^2 - |z|^2}{(R\zeta - z)^{2n}} \right| \le \frac{R^2 - |z|^2}{(R - |z|)^{2n}} \le \frac{R + |z|}{(R - |z|)^{2n-1}} \le \frac{2}{(R - r)^{2n-1}}$$

for any $\zeta \in S$ and $z \in rB$. Therefore, by subharmonicity of $|f(z)|^p$

$$|f(z)|^p \le \int_S \frac{R^2 - |z|^2}{|R\zeta - z|^{2n}} |f(R\zeta)|^p \, d\sigma(\zeta) \le \frac{2}{(R - r)^{2n-1}} \int_S |f(R\zeta)|^p \, d\sigma(\zeta). \tag{4}$$

As $M(R) = \int_S |f(R\zeta)|^p \, d\sigma(\zeta)$ is a nondecreasing function,

$$\begin{aligned} \int_R^1 |d\omega(t^2)| \int_S |f(R\zeta)|^p \, d\sigma(\zeta) &\le \int_R^1 \left(\int_S |f(t\zeta)|^p \, d\sigma(\zeta) \right) |d\omega(t^2)| \\ &= \int_{R < |w| < 1} |f(w)|^p |d\mu_\omega(t^2)| \le \|f\|^p_{p,\omega}. \end{aligned}$$

Consequently, by (4)

$$|f(z)| \le 2^{1/p} \|f\|_{p,\omega} (R - r)^{-(2n-1)/p} \left(\int_R^1 |d\omega(t^2)| \right)^{-1/p}.$$

Taking here $\max_{z \in K}$ we come to (3). □

Proposition 2. *For any $0 < p < \infty$, A^p_ω is closed subset of L^p_ω.*

Proof. Suppose $\|f_j - f\|_{p,\omega} \to 0$ as $j \to \infty$, where f_j is a sequence in A^p_ω and $f \in L^p_\omega$. We must show, that with respect to μ_ω f is equivalent to a function, which is holomorphic on B.

Let $K \in B$ be compact. By (3), $\|f_j(z) - f_k(z)\| \le C\|f_j - f_k\|_{p,\omega}$ for all $z \in K$ and all j, k. Because f_j is the fundamental sequence in A^p_ω, this implies that f_j converges uniformly on compact subsets of B to a function h that is holomorphic on B.

Because $f_j \to f$ in L^p_ω, by theorem of Riesz some subsequence of f_j converges to f pointwise almost everywhere with respect to μ_ω on B. It follows that $f = h$ almost everywhere on B, and $f \in A^p_\omega$, as desired. □

Corollary. *A^p_ω is a Banach space for $1 \le p < \infty$ and a complete metrical space for $0 < p < 1$.*

2 A representation of A^2_ω over the sphere

We start by

Proposition 3. *Let $\widetilde{\omega} \in \Omega_A$ be continuously differentiable in $[0,1)$ and such that $\widetilde{\omega}(t) \searrow$, $\widetilde{\omega}(1) = 0$ and $\widetilde{\omega}(0) = 1$. Further, let ω be the Volterra square of $\widetilde{\omega}$, i.e.*

$$\omega(x) = -\int_x^1 \widetilde{\omega}\left(\frac{x}{\sigma}\right) d\widetilde{\omega}(\sigma), \qquad 0 < x < 1. \tag{5}$$

Then $\omega \in \Omega$ and

$$\Delta_m(\omega) = \left[\Delta_m(\widetilde{\omega})\right]^2, \quad m \ge 0. \tag{6}$$

Proof. From (5) it follows that $\omega(1) = 0$. Besides, one can verify that

$$\omega'(x) = -\int_x^1 \widetilde{\omega}'(t)\widetilde{\omega}'\left(\frac{x}{t}\right)\frac{dt}{t} \le 0 \quad (0 < x < 1) \quad \text{and}$$

$$-\int_0^1 x^m d\omega(x) = \int_0^1 x^m dx \int_0^1 \widetilde{\omega}'(t)\widetilde{\omega}'\left(\frac{x}{t}\right)\frac{dt}{t} = \left[-\int_0^1 t^m d\widetilde{\omega}(t)\right]^2, \; m = 0, 1, \ldots$$

Thus, $\Delta_m(\omega) = \left[\Delta_m(\widetilde{\omega})\right]^2$ $(m \ge 0)$ and $\bigvee_0^1 \omega = \left(\bigvee_0^1 \widetilde{\omega}\right)^2 = 1$. Hence $\omega(0) = \omega(0) - \omega(1) = \bigvee_0^1 \omega = 1$, and for any $\delta \in [0,1)$

$$\begin{aligned}
\bigvee_\delta^1 \omega &= -\int_\delta^1 \omega'(x)dx = \int_\delta^1 dx \int_x^1 \widetilde{\omega}'(t)\widetilde{\omega}'\left(\frac{x}{t}\right)\frac{dt}{t} \\
&= \int_\delta^1 \widetilde{\omega}'(t)\frac{dt}{t}\int_\delta^t \widetilde{\omega}'\left(\frac{x}{t}\right)dx = \int_\delta^1 \widetilde{\omega}'(t)dt \int_{\delta/t}^1 \widetilde{\omega}'(\tau)d\tau \\
&\ge \int_{(\delta+1)/2}^1 \widetilde{\omega}'(t)dt \int_{2\delta/(1+\delta)}^1 \widetilde{\omega}'(\tau)d\tau = \bigvee_{(\delta+1)/2}^1 \widetilde{\omega} \bigvee_{2\delta/(1+\delta)}^1 \widetilde{\omega} > 0.
\end{aligned}$$

□

Theorem 1. *If* $f \in A^2_\omega$, *then the function*

$$\varphi_0(z) = L_{\widetilde{\omega}} f(z) = -\int_0^1 f(tz)\, d\widetilde{\omega}(t)$$

belongs to the ordinary Hardy space H^2 *in the unit ball, and*

$$f(z) = \int_S \varphi_0(\zeta) C_{\widetilde{\omega}}(z,\zeta)\, d\sigma(\zeta).$$

Besides

$$\|\varphi_0\|_{H^2} = \|f\|_{2,\omega}. \tag{7}$$

Proof. Let $f \in A^2_\omega$ and $f(w) = \sum a_k w^k$ be its power expansion. Then for any $r \in (0,1)$

$$\int_{rB} |f(w)|^2\, d\mu_\omega(w) = \int_0^r d\omega(\varrho^2) \left[\sum_k a_k (\varrho\zeta)^k \sum_s \overline{a}_s (\varrho\overline{\zeta})^s \right] d\sigma(\zeta)$$
$$= \sum_k \sum_s a_k \overline{a}_s \int_0^r \varrho^{|k|+|s|}\, d\omega(\varrho^2) \int_S \zeta^k \overline{\zeta}^s\, d\sigma(\zeta).$$

Taking in account that

$$\int_S \zeta^k \overline{\zeta}^s\, d\sigma(\zeta) = \begin{cases} \gamma_k & \text{if} \quad s = k \\ 0 & \text{if} \quad s \neq k. \end{cases}$$

(see [5], Propositions 1.4.8 and 1.4.9), and letting $r \to 1-0$ we get

$$\int_B |f(w)|^2\, d\mu_\omega(w) = \sum_k \gamma_k |a_k|^2 \Delta_{|k|}(\omega). \tag{8}$$

If $b_k = a_k \sqrt{\Delta_{|k|}(\omega)}$, then (8) becomes

$$\|f\|^2_{2,\omega} = \sum_k \gamma_k |b_k|^2, \tag{9}$$

and it is evident that $\varphi_0(z) = \sum_k b_k z^k$ belongs to H^2 in B. Consequently (see [5]), $\varphi_0(z)$ has the K-limit $\varphi_0(\zeta) = (K\text{-}\lim \varphi_0)(\zeta)$ almost everywhere on S, $\varphi_0(\zeta) \in L^2(S)$ and moreover

$$b_k = \frac{1}{\gamma_k} \int_S \overline{\zeta}^k \varphi_0(\zeta)\, d\sigma(\zeta), \qquad \|\varphi_0\|^2_{H^2} = \sum_k \gamma_k |b_k|^2.$$

Hence $\|\varphi_0\|_{H^2} = \|f\|_{2,\omega}$ by (9). Further, observe that in virtue of (6) and (1)

$$f(z) = \sum_k a_k z^k = \sum_k \frac{b_k z^k}{\sqrt{\Delta_{|k|}(\omega)}} = \sum_k \frac{z^k}{\gamma_k \sqrt{\Delta_{|k|}(\omega)}} \int_S \overline{\zeta}^k \varphi_0(\zeta)\, d\sigma(\zeta)$$
$$= \int_S \varphi_0(\zeta) \left[\sum_k \frac{z^k \overline{\zeta}^k}{\gamma_k \Delta_{|k|}(\widetilde{\omega})} \right] d\sigma(\zeta) = \int_S \varphi_0(\zeta) C_{\widetilde{\omega}}(z,\zeta)\, d\sigma(\zeta).$$

And it is evident that

$$\begin{aligned}\varphi_0(z) = \sum_k b_k z^k &= \sum_k a_k \sqrt{\Delta_{|k|}(\omega)} z^k = \sum_k a_k \Delta_{|k|}(\widetilde{\omega}) z^k \\ &= \sum_k a_k z^k \left(\int_0^1 t^{|k|} \, d\widetilde{\omega}(t) \right) = - \int_0^1 f(tz) \, d\widetilde{\omega}(t) = L_{\widetilde{\omega}} f(z).\end{aligned}$$

□

Following [2], for A^p_ω we shall use the notation $H^p(\alpha)$ in the special case

$$\omega(x) = \int_x^1 t^{n-1}(1-t)^\alpha \, dt \qquad (\alpha > -1) \tag{10}$$

In this case (2) becomes

$$\|f\|_{H^p(\alpha)} = \left\{ \int_B (1-|w|^2)^\alpha |f(w)|^p \, d\nu(w) \right\}^{1/p} < +\infty$$

and Theorem 1, in essence, becomes the multidimensional analog of Theorem V in [2], the case $\alpha = 0$ of which is due to M.V.Keldysch (see [2]).

Theorem 2. *Let* $f(z) \in H^2(\alpha)$ $(\alpha > -1)$. *Then the function*

$$\varphi_0(z) = \frac{\Gamma(n + \frac{\alpha+1}{2})}{\Gamma(n)\Gamma(\frac{\alpha+1}{2})} \int_0^1 f(tz) t^{n-1} (1-t)^{\frac{\alpha-1}{2}} \, dt$$

belongs to $H^2(B)$ *and the following integral representation is true:*

$$f(z) = \int_S \frac{\varphi_0(\zeta) \, d\sigma(\zeta)}{(1 - \langle z, \zeta \rangle)^{n + \frac{\alpha+1}{2}}}.$$

Proof. One can verify that

$$\widetilde{\omega}(x) = \frac{\Gamma(n + \frac{\alpha+1}{2})}{\Gamma(n)\Gamma(\frac{\alpha+1}{2})} \int_x^1 t^{n-1} (1-t)^{\frac{\alpha-1}{2}} \, dt.$$

satisfies the requirements of Theorem 1. One can calculate, that the corresponding kernel is of the form

$$C_{\widetilde{\omega}}(z, \zeta) = \frac{1}{(1 - \langle z, \zeta \rangle)^{n + \frac{\alpha+1}{2}}}.$$

Similar to [3], one can prove that the Volterra square of $\widetilde{\omega}(x)$ satisfies $\omega'(x) \asymp (1-x)^\alpha$. This means that $A^2_\omega = H^2(\alpha)$ in the considered case. □

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