ON EVERYWHERE DIVERGENCE OF THE STRONG Φ-MEANS OF WALSH-FOURIER SERIES

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Abstract. Almost everywhere strong exponential summability of Fourier series in Walsh and trigonometric systems established by Rodin in 1990. We prove, that if the growth of a function $\Phi(t) : [0,\infty) \to [0,\infty)$ is bigger than the exponent, then the strong Φ-summability of a Walsh-Fourier series can fail everywhere. The analogous theorem for trigonometric system was proved before by one of the author of this paper.

1. Introduction

In the study of almost everywhere convergence and summability of Fourier series the trigonometric and Walsh systems have many common properties. Kolmogorov [9] in 1926 gave a first example of everywhere divergent trigonometric Fourier series. Existence of almost everywhere divergent Walsh-Fourier series first proved by Stein [15]. Then Schipp in [16] constructed an example of everywhere divergent Walsh-Fourier series. A significant complement to these divergence theorems are investigations on almost everywhere summability of Fourier series.

Let $\Phi(t) : [0,\infty) \to [0,\infty)$, $\Phi(0) = 0$, be an increasing continuous function. A numerical series with partial sums $s_1, s_2, \dots$ is said to be (strong) Φ-summable to a number s, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Phi\left(|s_k - s|\right) = 0. \tag{1.1}$$

We note that the condition (1.1) is as strong as rapidly growing is Φ, and in the case of $\Phi(t) = t^p$, $p > 0$, the condition (1.1) coincides with H^p-summability, well known in the theory of Fourier series. Marcinkiewicz-Zygmund in [11], [21] established almost everywhere H^p-summability for arbitrary trigonometric Fourier series (ordinary and conjugate). K. I. Oskolkov in [12] proved a.e. Φ-summability for trigonometric Fourier series if $\Phi(t) = O(t/\log\log t)$. Then V. Rodin [13] established the analogous with Φ satisfying the condition

$$\limsup_{t\to\infty} \frac{\log \Phi(t)}{t} < \infty, \tag{1.2}$$

which is equivalent to the bound $\Phi(t) < \exp(Ct)$ with some $C > 0$. Moreover, Rodin invented an interesting property, that is almost everywhere BMO-boundedness of Fourier series, and the a.e. Φ-summability immediately follows from this results,

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applying John-Nirenberg theorem. G. A. Karagulyan in [6, 7] proved that the condition (1.2) is sharp for a.e. Φ-summability for Fourier series. That is if

$$\limsup_{t\to\infty}\frac{\log\Phi(t)}{t}=\infty, \tag{1.3}$$

then there exists an integrable function $f\in L^1(0,2\pi)$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\Phi(|S_k(x,f)|)=\infty,\quad \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\Phi(|\tilde{S}_k(x,f)|)=\infty,$$

holds everywhere on $\mathbb{R}$, where $S_k(x,f)$ and $\tilde{S}_k(x,f)$ are the ordinary and conjugate partial sums of Fourier series of $f(x)$.

Analogous problems are considered also for Walsh series. Almost everywhere H^p-summability of Walsh-Fourier series with $p>0$ proved by F. Schipp in [17]. Almost everywhere Φ-summability with condition (1.2) proved by V. Rodin [14].

Theorem (Rodin)**.** *If $\Phi(t):[0,\infty)\to[0,\infty)$, $\Phi(0)=0$, is an increasing continuous function satisfying (1.2), then the partial sums of Walsh-Fourier series of any function $f\in L^1[0,1)$ satisfy the condition*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\Phi\left(|S_k(x,f)-f(x)|\right)=0$$

almost everywhere on $[0,1)$.

In this theorem and everywhere bellow the notation $S_k(x,f)$ stands for the partial sums of Walsh-Fourier series of $f\in L^1[0,1)$. In the present paper we establish, that, as in trigonometric case [7], the bound (1.2) is sharp for a.e. Φ-summability of Walsh-Fourier series. Moreover, we prove

Theorem. *If an increasing function $\Phi(t):[0,\infty)\to[0,\infty)$ satisfies the condition (1.3), then there exists a function $f\in L^1[0,1)$ such that*

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\Phi\left(|S_k(x,f)|\right)=\infty \tag{1.4}$$

holds everywhere on $[0,1)$.

It is clear this theorem generalizes Schipp's theorem on everywhere divergence of Walsh-Fourier series. S. V. Bochkarev in [1] has constructed a function $f\in L^1[0,1)$ such that

$$\limsup_{n\to\infty}\frac{|S_n(x,f)|}{\omega_n}=\infty \tag{1.5}$$

everywhere on $[0,1)$, where $\omega_n=o(\sqrt{\log n})$. It is easy to observe, that this theorem implies the existence of a function $f\in L^1[0,1)$ satisfying (1.4) whenever

$$\limsup_{t\to\infty}\frac{\log\Phi(t)}{t^2}=\infty,$$

instead of the condition (1.3), which was the best bound before. A divergence theorem like (1.5) with $\omega_n=o(\sqrt{\log n/\log\log n})$ for the trigonometric Fourier series established by S. V. Konyagin in [10].

We note also, that the problem of uniformly Φ-summability of trigonometric Fourier series, when $f(x)$ is a continuous function considered by V. Totik [19, 20].

He proved that the condition (1.2) is necessary and sufficient for uniformly Φ-summability of Fourier series of continuous functions. For the Walsh series the analogous problem is considered by S. Fridli and F. Schipp [4, 5], V. Rodin [14], U. Goginava and L. Gogoladze [2].

2. Proof of theorem

Recall the definitions of Rademacher and Walsh functions (see [3] or [18]). We consider the function

$$r_0(x) = \begin{cases} 1, & \text{if} \quad x \in [0, 1/2), \\ -1, & \text{if} \quad x \in [1/2, 1), \end{cases}$$

periodically continued over the real line. The Rademacher functions are defined by $r_k(x) = r_0(2^k x)$, $k = 0, 1, 2, \ldots$. Walsh system is obtained by all possible products of Rademacher functions. We shall consider the Paley ordering of Walsh system. We set $w_0(x) \equiv 1$. To define $w_n(x)$ when $n \geq 1$ we write n in dyadic form

$$n = \sum_{j=0}^{k} \varepsilon_j 2^j, \tag{2.1}$$

where $\varepsilon_k = 1$ and $\varepsilon_j = 0$ or 1 if $j = 0, 1, \ldots, k-1$, and set

$$w_n(x) = \prod_{j=0}^{k} (r_j(x))^{\varepsilon_j}.$$

The partial sums of Walsh-Fourier series of a function $f \in L^1[0, 1)$ have a formula

$$S_n(x, f) = \int_0^1 f(t) D_n(x \oplus t) dt,$$

where $D_n(x)$ is the Dirichlet kernel and $\oplus$ denotes the dyadic addition. We note that

$$D_{2^k}(x) = \begin{cases} 2^k, & \text{if} \quad x \in [0, 2^{-k}), \\ 0, & \text{if} \quad x \in [2^{-k}, 1). \end{cases}$$

Dirichlet kernel can be expressed by modified Dirichlet kernel $D_n^*(x)$ by

$$D_n(x) = w_n(x) D_n^*(x).$$

If $n \in \mathbb{N}$ has the form (2.1), then we have

$$D_n^*(x) = \sum_{j=0}^{k} \varepsilon_j D_{2^j}^*(x) = \sum_{j=0}^{k} \varepsilon_j r_j(x) D_{2^j}(x).$$

We shall write $a \lesssim b$, if $a < c \cdot b$ and $c > 0$ is an absolute constant. The notation $\mathbb{I}_E$ stands for the indicator function of a set E. An interval is said to be a set of the form $[a, b)$. For a dyadic interval δ we denote by δ^+ and δ^- left and right halves of δ. We denote the spectrum of a Walsh polynomial $P(x) = \sum_{k=0}^{m} a_k w_k(x)$ by

$$\operatorname{sp} P(x) = \{k \in \mathbb{N} \cup 0 : \ a_k \neq 0\}.$$

In the proof of following lemma we use a well known inequality

$$\left| \left\{ x \in (0, 1) : \left| \sum_{k=1}^{n} a_k r_k(x) \right| \leq \lambda \right\} \right| \geq 1 - 2 \exp\left(-\lambda^2 / 4 \sum_{k=1}^{n} a_k^2 \right), \quad \lambda > 0, \tag{2.2}$$

for Rademacher polynomials (see for example [8], chap. 2, theorem 5).

Lemma 1. *If $n \in \mathbb{N}$, $n > 50$, then there exists a set $E_n \subset [0,1)$, which is a union of some dyadic intervals of the length 2^{-n}, satisfies the inequality*

$$|E_n| > 1 - 2\exp(-n/36), \tag{2.3}$$

and for any $x \in E_n$ there exists an integer $m = m(x) < 2^n$ such that

$$\int_0^x D_m^*(x \oplus t)dt \geq \frac{n}{30}. \tag{2.4}$$

Proof. We define

$$E_n = \left\{ x \in [0,1) : \left| \sum_{j=1}^{n} r_j(x) r_{j+1}(x) \right| < \frac{n}{3} \right\} \tag{2.5}$$

Since $\phi_j(x) = r_j(x) r_{j+1}(x)$, $j = 1, 2, \ldots, n$ are independent functions, taking values ± 1 equally, the inequality (2.2) holds for $\phi_j(x)$ functions too. Applying (2.2) in (2.5) we will get the bound (2.3). Observe that for a fixed $x \in E_n$ we have

$$\#\{j \in \mathbb{N} : 1 \leq j \leq n : r_j(x) r_{j+1}(x) = -1\} > n/3, \tag{2.6}$$

where $\#A$ denotes the cardinality of a set A. On the other hand the value in (2.6) characterizes the number of sign changes in the sequence $r_1(x), r_2(x), \ldots, r_{n+1}(x)$. Using this fact, we may fix integers $1 \leq k_1 < k_2 < \ldots < k_\nu \leq n$, such that

$$r_{k_i}(x) = 1, \quad r_{k_i+1}(x) = -1, \quad i = 1, 2, \ldots, \nu, \quad \nu \geq \frac{n}{6} - 1. \tag{2.7}$$

Suppose δ_j is the dyadic interval of the length 2^{-j} containing the point x. Observe that (2.7) is equivalent to the condition

$$x \in \left((\delta_{k_j})^+ \right)^- . \tag{2.8}$$

This implies

$$\left((\delta_{k_j})^+ \right)^+ \subset [0, x), \tag{2.9}$$

$$r_{k_j}(x \oplus t) = 1, \quad t \in \delta_{k_j} \cap [0, x). \tag{2.10}$$

Now consider the integer

$$m = 2^{k_1} + 2^{k_2} + \ldots + 2^{k_\nu}.$$

Using (2.9) and (2.10), we obtain

$$\begin{aligned}
\int_0^x D_m^*(x \oplus t)dt &= \sum_{j=1}^{\nu} \int_0^x r_{k_j}(x \oplus t) D_{2^{k_j}}(x \oplus t)dt \\
&= \sum_{j=1}^{\nu} 2^{k_j} \int_{\delta_{k_j} \cap [0,x)} r_{k_j}(x \oplus t)dt \\
&\geq \sum_{j=1}^{\nu} 2^{k_j} \int_{\left((\delta_{k_j})^+\right)^+} r_{k_j}(x \oplus t)dt \\
&= \sum_{j=1}^{\nu} 2^{k_j - 2} |\delta_{k_j}| = \frac{\nu}{4} > \frac{n}{30}.
\end{aligned}$$

$\square$

Lemma 2. *For any integer $n > n_0$ there exists a Walsh polynomial $f(x) = f_n(x)$ such that*

$$\|f\|_1 \le 4, \quad \operatorname{sp} f(x) \subset [p(n), q(n)], \tag{2.11}$$

$$\sup_{N\in[p(n),2q(n)]} \frac{\#\{k \in \mathbb{N} : 1 \le k \le N,\, |S_k(x, f)| > n/40\}}{N} \gtrsim 2^{-2n}, \tag{2.12}$$

where $p(n)$, $q(n)$ are positive integers, and n_0 is an absolute constant.

Proof. We define

$$\theta_k = \frac{k-1}{2^n} + \frac{k-1}{4^n} \in \Delta_k = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right), \quad k = 1, 2, \dots, 2^n.$$

Let E_n be the set obtained in Lemma 1. We define $f(x)$ by

$$f(x) = 2^\gamma \cdot \mathbb{I}_{(E_n)^c}(x) r_n(x) + \frac{1}{2^n} \sum_{j=1}^{2^n} \Big(D_{u_{2^n}}(x \oplus \theta_j) - D_{u_j}(x \oplus \theta_j) \Big), \tag{2.13}$$

where

$$\gamma = \left[\log_2(\exp(n/36)) \right], \tag{2.14}$$

$$u_j = 2^{10(j+n)}, \quad j = 1, 2, \dots, 2^n. \tag{2.15}$$

We have

$$\begin{aligned} &\operatorname{sp} \left(\mathbb{I}_{(E_n)^c}(x) r_n(x)\right) \subset [2^n, 2^{n+1}), \\ &\operatorname{sp} \left(D_{u_{2^n}}(x \oplus \theta_j) - D_{u_j}(x \oplus \theta_j)\right) \subset (u_j, u_{2^n}] \subset [2^n, u_{2^n}], \end{aligned}$$

and therefore

$$\operatorname{sp} f(x) \subset [p(n), q(n)], \quad p(n) = 2^n, \quad q(n) = u_{2^n}.$$

Using (2.3) and (2.14), we obtain

$$\|f\|_1 \le 2^\gamma (1 - |E_n|) + 2 \le \exp(n/36) \cdot 2 \exp(-n/36) + 2 = 4.$$

From the expression (2.13) it follows that any value taken by $f(x)$ is either 0 or a sum of different numbers of the form $\pm 2^k$ with $k \ge \gamma$. This implies

$$|f(x)| \ge 2^\gamma \ge \frac{\exp(n/36)}{2} > \frac{n}{40}, \quad n > n_0 = 150,$$

whenever

$$x \in \operatorname{supp} f = (E_n)^c \bigcup \left(\bigcup_{j=1}^{2^n - 1} (\theta_j \oplus \operatorname{supp} D_{u_j}) \right). \tag{2.16}$$

On the other hand if $l \ge q(n)$ and x satisfies (2.16), then we have

$$|S_l(x, f)| = |f(x)| > \frac{n}{40}.$$

Thus we obtain

$$\frac{\#\{k \in \mathbb{N} : 1 \le k \le 2q(n),\, |S_k(x, f)| > n/40\}}{2q(n)} \ge \frac{1}{2} > 2^{-2n},$$

which implies (2.12). Now consider the case when (2.16) doesn't hold. We may suppose that

$$x \in \Delta_k \setminus \operatorname{supp} f, \quad 1 \le k \le 2^n. \tag{2.17}$$

According to Lemma 1, there exists an integer $m = m(x) < 2^n$ satisfying one of the inequality (2.4). First we suppose it satisfies the first one. Together with m we consider

$$p = p(x) = m(x)(1 + 2^n) < 2^{2n}.$$

Using the definition of θ_j, observe, that

$$\begin{aligned} w_m(\theta_k) &= w_m\left(\frac{k-1}{2^n}\right), \\ w_{m\cdot 2^n}(\theta_k) &= w_{m\cdot 2^n}\left(\frac{k-1}{4^n}\right) = w_m\left(\frac{k-1}{2^n}\right), \end{aligned}$$

and therefore we get

$$w_p(\theta_k) = w_m(\theta_k) w_{m\cdot 2^n}(\theta_k) = 1, \quad k = 1, 2, \ldots, 2^n. \tag{2.18}$$

Define

$$L(x) = \{l \in \mathbb{N} : \, l = p + \mu \cdot 2^{2n}, \mu \in \mathbb{N}\}. \tag{2.19}$$

Once again using the definition of θ_k as well as (2.18), we conclude

$$w_l(\theta_k) = w_p(\theta_k) w_{\mu\cdot 2^{2n}}(\theta_k) = 1, \quad k = 1, 2, \ldots, 2^n, \quad l \in L(x). \tag{2.20}$$

Suppose

$$l \in L(x) \cap [u_{k-1}, u_k), \quad k \le 2^n. \tag{2.21}$$

Since x is taken outside of $\operatorname{supp} f$, we have

$$\begin{aligned} S_l(x, f) &= \frac{1}{2^n}\left(\sum_{j=1}^{k-1} D_l(x \oplus \theta_j) - \sum_{j=1}^{k-1} D_{u_j}(x \oplus \theta_j)\right) \\ &= \frac{1}{2^n}\sum_{j=1}^{k-1} D_l(x \oplus \theta_j). \end{aligned} \tag{2.22}$$

On the other hand by (2.20) we get

$$\begin{aligned} \frac{1}{2^n}\left|\sum_{j=1}^{k-1} D_l(x \oplus \theta_j)\right| &= \frac{1}{2^n}\left|\sum_{j=1}^{k-1} w_l\left(\theta_j\right) D_l^*(x \oplus \theta_j)\right| \\ &= \frac{1}{2^n}\left|\sum_{j=1}^{k-1} D_l^*(x \oplus \theta_j)\right|. \end{aligned} \tag{2.23}$$

Using the definition of $D_l^*(x)$, observe that

$$D_l^*(x) = D_p^*(x) + D_{\mu\cdot 2^{2n}}^*(x) = D_m^*(x) + D_{m\cdot 2^n}^*(x) + D_{\mu\cdot 2^{2n}}^*(x).$$

Since the supports of the functions $D_{m\cdot 2^n}^*(t)$ and $D_{\mu\cdot 2^{2n}}^*(t)$ are in Δ_1, we conclude

$$D_l^*(x \oplus \theta_j) = D_m^*(x \oplus \theta_j), \quad x \in \Delta_k, \quad j \ne k. \tag{2.24}$$

Thus, applying Lemma 1 and (2.23), we obtain the bound

(2.25)
$$\frac{1}{2^n}\left|\sum_{j=1}^{k-1} D_l(x\oplus\theta_j)\right| = \frac{1}{2^n}\left|\sum_{j=1}^{k-1} D_m^*(x\oplus\theta_j)\right|$$
$$\geq \int_0^x D_m^*(x\oplus t)dt - 1 > \frac{n}{30} - 1 > \frac{n}{40}, \quad n > n_0 = 150,$$

which holds whenever l satisfies (2.21). Taking into account of (2.22) and (2.25), we get

$$\frac{\#\{l\in\mathbb{N}:\ 1\leq l\leq u_k,\ |S_l(x,f)|>n/40\}}{u_k} \geq \frac{\#\left(L(x)\cap[u_{k-1},u_k)\right)}{u_k} \gtrsim 2^{-2n},$$

which completes the proof of lemma. □

Proof of theorem. We may choose numbers $\{n_k\}_{k=1}^\infty$ and $\{\alpha_k\}_{k=1}^\infty$ such that

$$p(n_{k+1}) > 2q(n_k), \tag{2.26}$$

$$\Phi\left(\frac{n_k}{50\cdot 2^k}\right) > \exp(2n_k), \tag{2.27}$$

$$n_{k+1} > 800k2^k q(n_k), \tag{2.28}$$

where $p(n)$ and $q(n)$ are the sequences determined in Lemma 2. We just note that (2.27) may guarantee by using (1.3). Applying Lemma 2, we get polynomials $g_k(x) = f_{n_k}(x)$, which satisfy (2.12) for any $x\in[0,1)$. We have

$$f(x) = \sum_{k=1}^\infty 2^{-k} g_k(x) \in L^1[0,1).$$

The condition (2.26) provides increasing spectrums of these polynomials. Thus, if $p(n_k) < l \leq q(n_k)$, then we have

$$\begin{aligned}|S_l(x,f)| &= \left|\sum_{j=1}^\infty 2^{-j} S_l(x,g_j)\right| = \left|\sum_{j=1}^{k-1} 2^{-j} g_j(x) + 2^{-k} S_l(x,g_k)\right|\\ &\geq 2^{-k}|S_l(x,g_k)| - 4(k-1)q(n_{k-1}).\end{aligned} \tag{2.29}$$

Applying Lemma 2, for any $x\in[0,1)$ we may find a number $N_k\in[p(n_k),2q(n_k)]$ such that

$$\#\{l\in\mathbb{N}:\ p(n_k) < l \leq N_k,\ |S_l(x,g_k)| > n_k/40\} \gtrsim \frac{N_k}{2^{2n_k}}.$$

Thus, using also (2.28) and (2.29), we conclude

$$\#\{l\in\mathbb{N}:\ p(n_k) < l \leq N_k,\ |S_l(x,f)| > n_k/50\cdot 2^k\} \gtrsim \frac{N_k}{2^{2n_k}}$$

and finally, using (2.27) we obtain

$$\frac{1}{N_k}\sum_{j=1}^{N_k}\Phi(|S_j(x,f)|) \gtrsim \frac{1}{N_k}\cdot\frac{N_k}{2^{2n_k}}\cdot\Phi\left(\frac{n_k}{50\cdot 2^k}\right) \geq \left(\frac{e}{2}\right)^{2n_k}, \quad k=1,2,\ldots.$$

This implies the divergence of Φ-means at a point $x\in[0,1)$ taken arbitrarily, which completes the proof of the theorem. □

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