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# On Generalizations of Fatou's Theorem for the Integrals with General Kernels

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**Abstract** We define $\lambda(r)$-convergence, which is a generalization of nontangential convergence in the unit disc. We prove Fatou-type theorems on almost everywhere nontangential convergence of Poisson–Stieltjes integrals for general kernels $\{\varphi_r\}$, forming an approximation of identity. We prove that the bound

$$\limsup_{r\to 1} \lambda(r)\|\varphi_r\|_\infty < \infty$$

is necessary and sufficient for almost everywhere $\lambda(r)$-convergence of the integrals

$$\int_{\mathbb{T}} \varphi_r(t-x)d\mu(t).$$

**Keywords** Fatou theorem · Littlewood theorem · Harmonic functions

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## 1 Introduction

In his famous paper Fatou [5] proved

**Theorem A** (Fatou, 1906) *Any bounded analytic function on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ has nontangential limit for almost all boundary points.*

**Theorem B** (Fatou, 1906) *If a function of bounded variation $\mu(t)$ is differentiable at $x_0 \in \mathbb{T}$, then the Poisson integral*

$$\mathcal{P}_r(x, d\mu) = \frac{1}{2\pi} \int\limits_{\mathbb{T}} \frac{1 - r^2}{1 - 2r\cos(x - t) + r^2} d\mu(t)$$

*converges non-tangentially to $\mu'(x_0)$ as $r \to 1$.*

These are two fundamental theorems, having many applications in different mathematical theories (analytic functions, Hardy spaces, harmonic analysis, differential equations, etc.). J. Littlewood [8] made an important complement to these theorems, proving essentiality of the nontangential approach in Fatou's theorems.

**Theorem C** (Littlewood, 1927) *Let $\gamma \subset \overline{D}$ be an arbitrary curve, internally tangent at $z = 1$ and having no other common point with $|z| = 1$. Let $\gamma_x$ be the rotation of $\gamma$ about the origin by $e^{ix}$. Then there exists a bounded analytic function $f(z)$, $z \in D$, which does not have boundary limit along $\gamma_x$ for almost every $e^{ix}$.*

There are various generalizations of these theorems in different aspects. Lohwater and Piranian [9] improved Littlewood's theorem, replacing almost everywhere divergence with everywhere. H. Aikawa in [1], [2] extended Littlewood's theorem for harmonic functions. Almost everywhere convergence over some semi-tangential regions investigated by Nagel and Stein [10], Di Biase [3], Di Biase–Stokolos–Svensson–Weiss [4]. P. Sjögren [15–17], J.-O. Rönning [11–13], V. G. Krotov [7], I. N. Katkovskaya and V. G. Krotov [6] obtained some tangential convergence properties for the square root Poisson integral. Unfortunately, we are not able to talk about the details of these investigations within this paper. Some of them will be discussed in the last section.

We define $\lambda(r)$-convergence, which is a generalization of nontangential convergence in the unit disc. Let $\lambda(r) : (0, 1) \to \mathbb{R}_+$ be a function with $\lambda(r) \searrow 0$ as $r \to 1$. For a given $x \in \mathbb{T}$ we define $\lambda(r, x)$ to be the interval $[x - \lambda(r), x + \lambda(r)]$. If $\lambda(r) \geq \pi$ we assume that $\lambda(r, x) = \mathbb{T}$. Let $F_r(x)$ be a family of functions from $L^1(\mathbb{T})$, where $r$ varies in $(0, 1)$. We say $F_r(x)$ is $\lambda(r)$-convergent at a point $x \in \mathbb{T}$ to a value $A$, if

$$\lim_{r\to 1} \sup_{\theta\in\lambda(r,x)} |F_r(\theta) - A| = 0.$$

Alternately this relation will be denoted by

$$\lim_{\substack{r\to 1 \\ \theta\in\lambda(r,x)}} F_r(\theta) = A.$$

It is clear that the non-tangential convergence in the unit disc is the case of $\lambda(r) = c(1-r)$.

Given a function of bounded variation, $\mu(t)$ defines a Borel measure on $\mathbb{T}$. We consider the family of integrals

$$\Phi_r(x, d\mu) = \int_{\mathbb{T}} \varphi_r(x-t)d\mu(t), \quad 0 < r < 1, \tag{1.1}$$

where kernels $\varphi_r(x) \in L^\infty(\mathbb{T})$ form an approximation of identity (AI), that is,

1. $\int_{\mathbb{T}} \varphi_r(t)dt \to 1$ as $r \to 1$,
2. $\varphi_r^*(x) = \sup_{|x|\le|t|\le\pi} |\varphi_r(t)| \to 0$ as $r \to 1, 0 < |x| < \pi$,
3. $\sup_{0<r<1} \int_{\mathbb{T}} \varphi_r^*(x) < \infty$.

If $\mu(t)$ is absolute continuous and $d\mu(t) = f(t)dt$, $f \in L^1(\mathbb{T})$, then (1.1) will be denoted by $\Phi_r(x, f)$. We shall prove that the condition

$$\limsup_{r\to 1} \lambda(r)\|\varphi_r\|_\infty < \infty$$

is necessary and sufficient for almost everywhere $\lambda(r)$-convergence of the integrals $\Phi_r(x, d\mu)$ as well as $\Phi_r(x, f)$, $f \in L^1(\mathbb{T})$. Moreover, we prove that convergence holds at any point where $\mu(t)$ is differentiable. An analogous necessary and sufficient condition will be established also for almost everywhere $\lambda(r)$-convergence of $\Phi_r(x, f)$ if $f \in L^\infty(\mathbb{T})$, and this condition looks like

$$\lim_{\tau\to 0}\left(\limsup_{\delta\to 0} \sup_{\tau<r<1} \int_{-\delta\lambda(r)}^{\delta\lambda(r)} \varphi_r(t)dt\right) = 0.$$

If $\varphi_r$ coincides with the Poisson kernel, then $\|\varphi_r\|_\infty = O(1/(1-r))$ and from this results we deduce Fatou's theorems. Other consequences will be discussed in the last section.

## 2 Fatou-Type Theorems: The Case of Bounded Measures

We denote by BV $(\mathbb{T})$ the functions of bounded variation on $\mathbb{T}$. We say that the given approximation of identity $\{\varphi_r(x)\}$ is regular if each $\varphi_r(x)$ is positive, decreasing on $[0, \pi]$, and increasing on $[-\pi, 0]$. In this case the property 3) is unnecessary, because it will immediately follow from 1).

**Theorem 2.1** *Let $\{\varphi_r\}$ be a regular AI and $\lambda(r)$ satisfies the condition*

$$\limsup_{r\to 1} \lambda(r)\|\varphi_r\|_\infty < \infty. \tag{2.1}$$

*If $\mu(t) \in$ BV $(\mathbb{T})$ is differentiable at $x_0$, then*

$$\lim_{\substack{r\to 1\\ x\in\lambda(r,x_0)}} \Phi_r\,(x, d\mu) = \mu'(x_0).$$

An analogous theorem holds as well in the non-regular case of kernels, but at this time the points where (1.1) converges satisfy the strong differentiability condition. We say a function of bounded variation is strong differentiable at $x_0 \in \mathbb{T}$, if there exists a number $c$ such that the variation of the function $\mu(x) - cx$ has zero derivative at $x_0$. If $d\mu(t) = f(t)dt$, then this property means that $x_0$ is a Lebesgue point for $f(x)$, that is,

$$\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} |f(x) - f(x_0)|dx = 0.$$

It is well known that strong differentiability at $x_0$ implies the existence of $\mu'(x_0)$, and any function of bounded variation is strong differentiable almost everywhere.

**Theorem 2.2** *Let $\{\varphi_r\}$ be an arbitrary AI and $\lambda(r)$ satisfies the condition (2.1). If $\mu(t) \in \mathrm{BV}\,(\mathbb{T})$ is strong differentiable at $x_0 \in \mathbb{T}$, then*

$$\lim_{\substack{r\to 1\\ x\in\lambda(r,x_0)}} \Phi_r\,(x, d\mu) = \mu'(x_0).$$

The following theorem implies the sharpness of the condition (2.1) in Theorems 2.1 and 2.2.

**Theorem 2.3** *If $\{\varphi_r(x) \geq 0\}$ is an AI and the function $\lambda(r)$ satisfies the condition*

$$\limsup_{r\to 1} \lambda(r)\|\varphi_r\|_\infty = \infty, \tag{2.2}$$

*then there exists a positive function $f \in L^1(\mathbb{T})$ such that*

$$\limsup_{\substack{r\to 1\\ y\in\lambda(r,x)}} \Phi_r\,(y,\, f) = \infty \tag{2.3}$$

*for all $x \in \mathbb{T}$.*

The following lemma plays a significant role in the proofs of Theorems 2.1 and 2.2.

**Lemma 2.4** *Let a positive function $\varphi(t) \in L^\infty(\mathbb{T})$ be decreasing on $[0, \pi]$ and increasing on $[-\pi, 0]$. Then for any numbers $\varepsilon \in (0, 1)$ and $\theta \in (-\pi, \pi)$ there exist a finite family of intervals $I_j \subset \mathbb{T}$, $j = 1, 2, \ldots, n$, containing 0 in their closures $\bar{I}_j$, and numbers $\varepsilon_j = \pm\varepsilon$ such that*

$$|I_j| \leq 2\sup\{|t| : \varphi(t) \geq \varepsilon\}, \quad j = 1, 2, \ldots, n,$$

$$\sum_{j=1}^{n} |I_j| < 10\varepsilon^{-1} \max\{1, |\theta| \cdot \|\varphi\|_\infty, \|\varphi\|_1\},$$

$$\left|\varphi(x-\theta) - \sum_{j=1}^{n} \varepsilon_j \mathbb{I}_{I_j}(x)\right| \leq \varepsilon.$$

*Proof* Denote

$$y_k = \sup\{t > 0 : \varphi(t) \geq \varepsilon k\},$$
$$x_k = \sup\{t > 0 : \varphi(-t) \geq \varepsilon k\}, \quad k = 1, 2, \ldots, l = \left[\frac{\|\varphi\|_\infty}{\varepsilon}\right].$$

Then we obviously have

$$y_0 = \pi, \quad 0 \leq y_l \leq y_{l-1} \leq \ldots \leq y_1 \leq \sup\{|t| : \varphi(t) \geq \varepsilon\}, \tag{2.4}$$
$$x_0 = \pi, \quad 0 \leq x_l \leq x_{l-1} \leq \ldots \leq x_1 \leq \sup\{|t| : \varphi(t) \geq \varepsilon\}, \tag{2.5}$$
$$\left|\varphi(x-\theta) - \varepsilon \sum_{k=1}^{l} \mathbb{I}_{(\theta - x_k, \theta + y_k)}(x)\right| \leq \varepsilon. \tag{2.6}$$

Without loss of generality we can suppose $0 \leq \theta < \pi$. Then we denote

$$k_0 = \max\{k : 0 \leq k \leq l, \ \theta - x_k \leq 0\}.$$

We define the desired intervals $I_j$, $j = 1, 2, \ldots, n = 2l - k_0$, by

$$I_j = \begin{cases} (\theta - x_j, \theta + y_j) & \text{if } j \leq k_0, \\ (0, \theta + y_j) & \text{if } k_0 < j \leq l, \\ (0, \theta - x_{j-l+k_0}] & \text{if } l < j \leq n = 2l - k_0. \end{cases}$$

Using the equality

$$\mathbb{I}_{(\theta - x_k, \theta + y_k)}(x) = \mathbb{I}_{(0, \theta + y_k)}(x) - \mathbb{I}_{(0, \theta - x_k]}(x) = \mathbb{I}_{I_k}(x) - \mathbb{I}_{I_{k+l-k_0}}(x), \quad k_0 < k \leq l,$$

we get

$$\varepsilon \sum_{k=1}^{l} \mathbb{I}_{(\theta - x_k, \theta + y_k)}(x) = \sum_{j=1}^{n} \varepsilon_j \mathbb{I}_{I_j}(x), \tag{2.7}$$

where

$$\varepsilon_j = \begin{cases} \varepsilon & \text{if } 1 \leq j \leq l, \\ -\varepsilon & \text{if } l < j \leq n. \end{cases} \tag{2.8}$$

We note that $\varepsilon_j = -\varepsilon$ in the case when $I_j$ coincides with one of the intervals $(0, \theta - x_k]$, $k_0 < k \leq l$. Hence we have

$$\sum_{j=l+1}^{n} |I_j| = \sum_{k=k_0+1}^{l} (\theta - x_k) \leq l \cdot \theta \leq \frac{\theta \|\varphi\|_\infty}{\varepsilon}. \tag{2.9}$$

From (2.6) and (2.7) we get

$$\left|\varphi(x-\theta)-\sum_{j=1}^{n}\varepsilon_j\mathbb{I}_{I_j}(x)\right|\le\varepsilon \tag{2.10}$$

and therefore by (2.8) we obtain

$$\left|\int_{\mathbb{T}}\varphi(t)dt-\varepsilon\sum_{j=1}^{l}|I_j|+\varepsilon\sum_{j=l+1}^{n}|I_j|\right|\le 2\pi\varepsilon<2\pi.$$

This and (2.9) imply

$$\varepsilon\sum_{j=1}^{n}|I_j|\le 2\varepsilon\sum_{j=l+1}^{n}|I_j|+\|\varphi\|_1+2\pi\le 2\theta\|\varphi\|_\infty+\|\varphi\|_1+2\pi,$$

which together with (2.4), (2.5), and (2.10) completes the proof of the lemma. □

*Proof of Theorem 2.1* Without loss of generality we may assume that $x_0=0$ and $\mu'(x_0)=0$. We fix a function $\theta(r):(0,1)\to\mathbb{R}$ with $|\theta(r)|\le\lambda(r)$. From (2.1) we get

$$|\theta(r)|\cdot\|\varphi_r\|_\infty\le M,\quad r_0<r<1. \tag{2.11}$$

Using the property 2) of the kernels $\{\varphi_r(t)\}$ we may define a collection of numbers $\varepsilon_r>0$ such that

$$\varepsilon_r\searrow 0,\quad \delta_r=\sup\{|t|:\varphi_r(t)\ge\varepsilon_r\}\to 0 \text{ as } r\to 1. \tag{2.12}$$

Applying Lemma 3.3, for any $0<r<1$ we define a family of intervals $\{I_j^{(r)},\ j=1,2,\dots,n_r\}$ such that

$$|I_j^{(r)}|\le 2\delta_r,\quad j=1,2,\dots,n_r, \tag{2.13}$$

$$\sum_{j=1}^{n_r}\left|I_j^{(r)}\right|<10(\varepsilon_r)^{-1}\max\{1,|\theta(r)|\cdot\|\varphi_r\|_\infty,\|\varphi_r\|_1\}, \tag{2.14}$$

$$\left|\varphi_r(\theta(r)-t)-\sum_{j=1}^{n_r}\varepsilon_j^{(r)}\mathbb{I}_{I_j^{(r)}}(t)\right|\le\varepsilon_r, \tag{2.15}$$

where $\varepsilon_j^{(r)}=\pm\varepsilon_r$. From (2.11) and (2.14) we conclude

$$\varepsilon_r\cdot\sum_{j=1}^{n_r}\left|I_j^{(r)}\right|\le L,\quad r_0<r<1, \tag{2.16}$$

where $L$ is a positive constant. From (2.12) and (2.15) we obtain

$$\Phi_r(\theta(r), d\mu) = \int_{\mathbb{T}} \varphi_r(\theta(r) - t) d\mu(t) = \sum_{j=1}^{n_r} \varepsilon_j^{(r)} \int_{I_j^{(r)}} d\mu(t) + o(1), \tag{2.17}$$

where $o(1) \to 0$ as $r \to 1$. Using this, we get

$$|\Phi_r(\theta(r), d\mu)| \leq \varepsilon_r \cdot \sum_{j=1}^{n_r} \left| I_j^{(r)} \right| \cdot \frac{1}{\left| I_j^{(r)} \right|} \left| \int_{I_j^{(r)}} d\mu(t) \right| + o(1).$$

According to (2.12) and (2.13), we have

$$\max_{1 \leq j \leq n_r} \frac{1}{\left| I_j^{(r)} \right|} \left| \int_{I_j^{(r)}} d\mu(t) \right| \to \mu'(0) = 0 \text{ as } r \to 1.$$

This together with (2.16) and (2.17) implies that $\Phi_r(\theta(r), d\mu) \to 0$ as $r \to 1$. □

*Proof of Theorem 2.2* Let $\theta(r)$ satisfy (2.11). We again assume that $x_0 = 0$, $\mu'(x_0) = 0$ and so we will have $|\mu|'(0) = 0$. Then, repeating the same process of the proof of Theorem 2.1 at this time for the functions $\varphi_r^*(t)$ together with the measure $|\mu|$, instead of (2.17) we obtain

$$\int_{\mathbb{T}} \varphi_r^* \left(\theta(r) - t\right) d|\mu|(t) = \sum_{j=1}^{n_r} \varepsilon_j^{(r)} \int_{I_j^{(r)}} d|\mu|(t) + o(1).$$

Then we get

$$|\Phi_r(\theta(r), d\mu)| \leq \int_{\mathbb{T}} \varphi_r^*(\theta(r) - t) d|\mu|(t) = \varepsilon_r \cdot \sum_{j=1}^{n_r} |I_j^{(r)}| \cdot \frac{1}{|I_j^{(r)}|} \int_{I_j^{(r)}} d|\mu|(t) + o(1).$$

Since $|\mu|(t)$ is differentiable at 0, we get

$$\Phi_r(\theta(r), d\mu) \to 0.$$

□

*Proof of Theorem 2.3* For any $0 < r < 1$ there exist a point $x_r \in (0, \pi)$, a number $0 < \delta_r < \lambda(r)/4$, and a measurable set $E_r$ such that

$$E_r \subset (x_r - \delta_r, x_r + \delta_r), \quad |E_r| > \frac{3\delta_r}{2}, \tag{2.18}$$

$$\varphi_r(x) > \frac{\|\varphi_r\|_\infty}{2}, \quad x \in E_r. \tag{2.19}$$

From these relations it follows that $\varphi_r^*(x) > \|\varphi_r\|_\infty/2$ if $x \in (0, |x_r|)$. On the other hand, by property 3) we have $\|\varphi_r^*\|_1 \le M$ for some constant $M > 0$. Thus we get

$$|x_r| \le \frac{2M}{\|\varphi_r\|_\infty}, \quad 0 < r < 1. \tag{2.20}$$

Denote

$$n(r) = \left[\frac{4\pi}{\lambda(r)}\right] \in \mathbb{N}, \tag{2.21}$$

$$\Delta_r = \bigcup_{k=0}^{n(r)-1} \left[\frac{2\pi k}{n(r)} - \delta_r, \frac{2\pi k}{n(r)} + \delta_r\right], \tag{2.22}$$

and consider the function

$$f_r(x) = \frac{\mathbb{I}_{\Delta_r(x)}}{|\Delta_r|} = \frac{\mathbb{I}_{\Delta_r}(x)}{2\delta_r n(r)}.$$

If $x \in \mathbb{T}$ is an arbitrary point, then

$$x \in \left[\frac{2\pi k_0}{n(r)}, \frac{2\pi(k_0+1)}{n(r)}\right)$$

for some $k_0$, $0 \le k_0 < n(r)$. Taking $\theta = x - x_r - 2\pi k_0/n(r)$, from (2.20) and (2.21) we obtain

$$|\theta| < \frac{2\pi}{n(r)} + |x_r| < \frac{2\pi\lambda(r)}{4\pi - \lambda(r)} + \frac{2M}{\|\varphi_r\|_\infty} = \lambda(r)\left(\frac{1}{2} + \frac{2M}{\lambda(r)\|\varphi_r\|_\infty}\right). \tag{2.23}$$

Using (2.2), we may fix a sequence $r_k \nearrow 1$ such that

$$\lambda(r_k)\|\varphi_{r_k}\|_\infty > M \cdot 4^k, \quad k = 1, 2, \ldots. \tag{2.24}$$

From (2.23) and (2.24) we conclude

$$|\theta| < \lambda(r), \text{ if } r = r_k. \tag{2.25}$$

Since $\varphi_r(x) \ge 0$, using (2.19) and (2.21), for the same $x$ we get

$$\Phi_r(x - \theta, f_r) = \int_{\mathbb{T}} \varphi_r(2\pi k_0/n(r) + x_r - t) f_r(t) dt$$

$$\geq \frac{1}{2\delta_r n(r)} \int\limits_{2\pi k_0/n(r)-\delta_r}^{2\pi k_0/n(r)+\delta_r} \varphi_r(2\pi k_0/n(r) + x_r - t)dt$$

$$= \frac{1}{2\delta_r n(r)} \int\limits_{x_r-\delta_r}^{x_r+\delta_r} \varphi_r(u)du$$

$$\geq \frac{1}{2\delta_r n(r)} \cdot \frac{3\delta_r}{2} \cdot \frac{\|\varphi_r\|_\infty}{2} \geq \frac{3\lambda(r)\|\varphi_r\|_\infty}{16}. \tag{2.26}$$

From (2.24), (2.25), and (2.26) we obtain

$$\sup_{\theta \in \lambda(r_k, x)} \Phi_{r_k}(\theta, f_{r_k}) \geq 3M \cdot 4^{k-2}, \quad x \in \mathbb{T}. \tag{2.27}$$

Using (2.24), we define

$$f(x) = \sum_{k=1}^{\infty} 2^{-k} f_{r_k}(x) \in L^1(\mathbb{T}).$$

From (2.27) and (2.24) it follows that

$$\sup_{\theta \in \lambda(r_k, x)} \Phi_{r_k}(\theta, f) \geq \sup_{\theta \in \lambda(r_k, x)} 2^{-k} \Phi_{r_k}(\theta, f_{r_k}) \geq 3M \cdot 2^{k-4},$$

which implies (2.3). □

## 3 Fatou-Type Theorems: The Case $L^\infty$

Let $\lambda(r) : (0, 1) \to \mathbb{R}$ be an arbitrary real function with $\lambda(r) \to 0$ as $r \to 1$. The quantity

$$\gamma_\lambda = \lim_{\tau \to 1} \left( \limsup_{\delta \to 0} \sup_{\tau < r < 1} \int\limits_{-\delta\lambda(r)}^{\delta\lambda(r)} \varphi_r(t)dt \right)$$

completely characterizes the almost everywhere $\lambda(r)$ convergence property of $\Phi_r(x, f)$ in $L^\infty(\mathbb{T})$.

**Theorem 3.1** *If $\{\varphi_r(x)\}$ is a regular AI consisting of even functions and $\gamma_\lambda = 0$, then for any $f \in L^\infty(\mathbb{T})$ the relation*

$$\lim_{\substack{r \to 1 \\ \theta \in \lambda(r,x)}} \Phi_r(\theta, f) = f(x)$$

*holds at any Lebesgue point.*

**Theorem 3.2** *If $\{\varphi_r(x)\}$ is a regular AI consisting of even functions and $\gamma_\lambda > 0$, then there exists a set $E \subset \mathbb{T}$, such that $\Phi_r(x, \mathbb{I}_E)$ is $\lambda(r)$-divergent at any $x \in \mathbb{T}$.*

Note that if $\lambda(t)$ satisfies the condition (2.1), then $\gamma_\lambda = 0$. One can easily construct a family of kernels $\{\varphi_r\}$ such that $\gamma_\lambda = 0$, but (2.1) is not satisfied. This means the $\gamma_\lambda = 0$ is a weaker condition than (2.1).

*Proof of Theorem 3.1* Since $\gamma_\lambda = 0$ for any $0 < \varepsilon < 1/2$ we may chose $\delta > 0$ and $0 < \tau < 1$, such that

$$\int_{-\delta\lambda(r)}^{\delta\lambda(r)} \varphi_r(t)dt < \varepsilon, \quad \tau < r < 1. \tag{3.1}$$

Then we define

$$\varphi_r^{(1)}(x) = \begin{cases} \varphi_r(x) - \varphi_r(\delta\lambda(r)) & \text{if } |x| \le \delta\lambda(r), \\ 0 & \text{if } \delta\lambda(r) < |x| < \pi, \end{cases}$$

and

$$\varphi_r^{(2)}(x) = \frac{\varphi_r(x) - \varphi_r^{(1)}(x)}{\|\varphi_r(x) - \varphi_r^{(1)}\|_{L^1}} = \frac{\varphi_r(x) - \varphi_r^{(1)}(x)}{1 - l_r}$$

where

$$l_r = \int_{-\delta\lambda(r)}^{\delta\lambda(r)} (\varphi_r(t) - \varphi_r(\delta\lambda(r))\, dt < \varepsilon < \frac{1}{2}, \quad \tau < r < 1.$$

It is clear, that $\{\varphi_r^{(2)}\}$ is a regular AI and we have

$$\varphi_r(x) = \varphi_r^{(1)}(x) + (1 - l_r)\varphi_r^{(2)}(x). \tag{3.2}$$

From (3.1) it follows that

$$\left| \int_{\mathbb{T}} \varphi_r^{(1)}(x-t) f(t) dt \right| \le \|f\|_\infty \int_{-\delta\lambda(r)}^{\delta\lambda(r)} \varphi_r(t)dt \le \varepsilon \|f\|_\infty \tag{3.3}$$

and

$$\varphi_r(\delta\lambda(r)) \cdot 2\delta\lambda(r) < \varepsilon, \quad \tau < r < 1.$$

Thus, using the definition of $\varphi_r^{(2)}(x)$, we get

$$\|\varphi_r^{(2)}\|_\infty \cdot \lambda(r) < \frac{\varepsilon}{2\delta(1 - l_r)} < \frac{\varepsilon}{4\delta}.$$

Using this and Theorem 2.1 we conclude that

$$\lim_{\substack{r\to 1\\ \theta\in\lambda(r,x)}} \int_{\mathbb{T}} \varphi_r^{(2)}(\theta - t) f(t)dt = f(x) \tag{3.4}$$

at any Lebesgue point. Now without loss of generality we assume that $f(x) \geq 0$. If $x$ is an arbitrary Lebesgue point, using (3.2), (3.3), and (3.4) we get

$$\limsup_{\substack{r\to 1\\ \theta\in\lambda(r,x)}} \Phi_r(\theta, f) \leq \varepsilon\|f\|_\infty + f(x),$$
$$\liminf_{\substack{r\to 1\\ \theta\in\lambda(r,x)}} \Phi_r(\theta, f) \geq -\varepsilon\|f\|_\infty + (1-\varepsilon)f(x).$$

Since $\varepsilon$ can be taken sufficiently small, we get

$$\lim_{\substack{r\to 1\\ \theta\in\lambda(r,x)}} \Phi_r(\theta, f) = f(x),$$

and the theorem is proved. $\square$

If $f(x)$ is a function defined on a set $E \subset \mathbb{T}$ we denote

$$\mathrm{OSC}_{x\in E}\, f(x) = \sup_{x,y\in E} |f(x) - f(y)|.$$

**Lemma 3.3** *Let*

$$U_n^\delta = \bigcup_{k=0}^{n-1} \left( \frac{\pi(2k+1-\delta)}{n}, \frac{\pi(2k+1+\delta)}{n} \right), \quad n \in \mathbb{N}, \quad 0 < \delta < \frac{1}{2},$$

*and $J \subset \mathbb{T}$, $\pi > |J| \geq 16\pi/n$, is an arbitrary closed interval. If a measurable set $E \subset \mathbb{T}$ satisfies either*

$$E \cap J = J \cap U_n^\delta \text{ or } E \cap J = J \setminus U_n^\delta,$$

*and $\varphi(x) \in L^\infty(\mathbb{T})$ is an even decreasing on $[0, \pi]$ function, then*

$$\mathrm{OSC}_{\theta\in\left[x-\frac{4\pi}{n}, x+\frac{4\pi}{n}\right]} \int_{\mathbb{T}} \varphi(\theta - t)\mathbb{I}_E(t)dt > \int_{-\frac{\pi\delta}{n}}^{\frac{\pi\delta}{n}} \varphi(t)dt - 16\delta - 2\pi\varphi\left(\frac{|J|}{4}\right), \tag{3.5}$$

*for any $x \in J$.*

*Proof* We suppose $J = [a, b]$ and

$$\frac{2\pi(p-1)}{n} < a \le \frac{2\pi p}{n}, \quad \frac{2\pi(q-1)}{n} < b \le \frac{2\pi q}{n}.$$

First we consider the case

$$E \cap J = J \cap U_n^\delta. \tag{3.6}$$

If $x \in J$, then

$$x \in I = \left[\frac{2\pi(m-1)}{n}, \frac{2\pi m}{n}\right]$$

for some $p \le m \le q$. Without loss of generality we may assume that the center of $I$ is on the left-hand side of the center of $J$. Then we will have

$$b - \frac{2\pi(m+1)}{n} \ge \frac{|J|}{2} - \frac{4\pi}{n} \ge \frac{|J|}{4}. \tag{3.7}$$

It is clear that the points

$$\theta_1 = \frac{2\pi m}{n} + \frac{\pi}{n}, \quad \theta_2 = \frac{2\pi(m+1)}{n} \tag{3.8}$$

are in the interval $[0, x + 4\pi/n]$. Besides, we have

$$\begin{aligned}
\int_{\mathbb{T}} \varphi(\theta_1 - t)\mathbb{I}_E(t)dt - \int_{\mathbb{T}} \varphi(\theta_2 - t)\mathbb{I}_E(t)dt \\
&= \int_{\theta_2-\pi}^{a} [\varphi(\theta_1 - t) - \varphi(\theta_2 - t)]\mathbb{I}_E(t)dt \\
&\quad + \int_{a}^{b} [\varphi(\theta_1 - t) - \varphi(\theta_2 - t)]\mathbb{I}_E(t)dt \\
&\quad + \int_{b}^{\theta_2+\pi} [\varphi(\theta_1 - t) - \varphi(\theta_2 - t)]\mathbb{I}_E(t)dt \\
&= A_1 + A_2 + A_3.
\end{aligned}$$

Since $\varphi$ is decreasing on $[0, \pi]$ we have

$$A_1 \ge 0. \tag{3.9}$$

If $t \in [b, \theta_2 + \pi]$ then, using (3.7), we get

$$t - \theta_2 \ge b - \theta_2 \ge \frac{|J|}{4}, \quad t - \theta_1 \ge \frac{|J|}{4},$$

which implies

$$|A_3| \leq 2\pi\varphi\left(\frac{|J|}{4}\right). \tag{3.10}$$

To estimate $A_2$ we denote

$$a_k = \int\limits_{\pi(k-\delta)/n}^{\pi(k+\delta)/n} \varphi(t)dt, \quad k \in \mathbb{Z}. \tag{3.11}$$

We have

$$a_0 = \int\limits_{-\pi\delta/n}^{\pi\delta/n} \varphi(t)dt. \tag{3.12}$$

Using properties of $\varphi$ we have $a_k = a_{-k}$ and $a_1 \geq a_2 \geq \ldots$. Using Chebyshev's inequality we have $\varphi(t) \leq 1/t$. Thus we obtain

$$a_k \leq a_1 = \int\limits_{\pi(1-\delta)/n}^{\pi(1+\delta)/n} \varphi(t)dt \leq \frac{2\pi\delta/n}{\pi(1-\delta)/n} = \frac{2\delta}{1-\delta} < 4\delta, \quad k \geq 1. \tag{3.13}$$

Using (3.6), (3.11), (3.12), and (3.13), we get

$$\begin{aligned}
A_2 &\geq \sum_{k=p}^{q-2} \int\limits_{\pi(2k+1-\delta)/n}^{\pi(2k+1+\delta)/n} [\varphi(\theta_1 - t) - \varphi(\theta_2 - t)]dt - 8\delta \\
&= \sum_{k=p}^{q-1} \int\limits_{\pi(2(m-k)-\delta)/n}^{\pi(2(m-k)+\delta)/n} \varphi(t)dt - \sum_{k=p}^{q-1} \int\limits_{\pi(2(m-k)+1-\delta)/n}^{\pi(2(m-k)+1+\delta)/n} \varphi(t)dt - 8\delta \\
&= \sum_{k=m-q+1}^{m-p} a_{2k} - \sum_{k=m-q+1}^{m-p} a_{2k+1} - 8\delta \geq a_0 - a_1 - a_{-1} - 8\delta \\
&> \int\limits_{-\pi\delta/n}^{\pi\delta/n} \varphi(t)dt - 16\delta.
\end{aligned}$$

Combining this with (3.9) and (3.10), we get

$$\int\limits_{\mathbb{T}} \varphi(\theta_1 - t)\mathbb{I}_E(t)dt - \int\limits_{\mathbb{T}} \varphi(\theta_2 - t)\mathbb{I}_E(t)dt \geq \int\limits_{-\frac{\pi\delta}{n}}^{\frac{\pi\delta}{n}} \varphi(t)dt - 16\delta - 2\pi\varphi\left(\frac{|J|}{4}\right),$$

which together with (3.8) implies (3.5). To deduce the case $E \cap J = J \setminus U_n^\delta$ notice that for the complement $E^c$ we have $E^c \cap J = J \cap U_n^\delta$ and so (3.5) holds for $E^c$. Therefore, we obtain

$$\begin{aligned}
&\mathrm{OSC}_{\theta\in\left[x-\frac{4\pi}{n},x+\frac{4\pi}{n}\right]}\int_{\mathbb{T}}\varphi(\theta-t)\mathbb{I}_E(t)dt\\
&\qquad=\mathrm{OSC}_{\theta\in\left[x-\frac{4\pi}{n},x+\frac{4\pi}{n}\right]}\left(\|\varphi\|_1-\int_{\mathbb{T}}\varphi(\theta-t)\mathbb{I}_E(t)dt\right)\\
&\qquad=\mathrm{OSC}_{\theta\in\left[x-\frac{4\pi}{n},x+\frac{4\pi}{n}\right]}\left(\int_{\mathbb{T}}\varphi(\theta-t)\mathbb{I}_{E^c}(t)dt\right)\\
&\qquad>\int_{-\frac{\pi\delta}{n}}^{\frac{\pi\delta}{n}}\varphi(t)dt-16\delta-2\pi\varphi\left(\frac{|J|}{4}\right),
\end{aligned}$$

which completes the proof of the lemma. □

*Proof of Theorem 3.2* Since $\gamma_\lambda > 0$, there exist sequences $\delta_k \searrow 0$ and $r_k \to 1$, such that

$$\int_{-\delta_k\lambda(r_k)}^{\delta_k\lambda(r_k)}\varphi_{r_k}(t)dt > \frac{\gamma_\lambda}{2}, \quad k=1,2,\ldots. \tag{3.14}$$

Denote

$$U_k = U_{n_k}^{\delta_k}, \quad n_k = \left[\frac{\pi}{\lambda(r_k)}\right], \tag{3.15}$$

where $U_n^\delta$ is defined in Lemma 3.3. Define the sequences of measurable sets $E_n$ by

$$E_1 = U_1, \quad E_k = E_{k-1} \triangle U_k = (E_{k-1}\setminus U_k)\cup(U_k\setminus E_{k-1}), \quad k>1.$$

We say $J$ is an adjacent interval for $E_k$ if it is a maximal interval containing either $E_k$ or $(E_k)^c$. The family of all these intervals form a covering of the whole of $\mathbb{T}$. It is easy to observe that a suitable selection of $\delta_k$ and $r_k$ may provide

$$\varphi_{r_k}\left(\frac{|J|}{4}\right) < \frac{\gamma_\lambda}{16\pi}, \text{ if } J \text{ is adjacent for } E_{k-1}, \tag{3.16}$$

$$\delta_j \le \frac{\gamma_\lambda}{2^{j+5}\|\varphi_{r_k}\|_\infty}, \quad j \ge k+1. \tag{3.17}$$

It is easy to observe that if $k < m$, then

$$\|\mathbb{I}_{E_k} - \mathbb{I}_{E_m}\|_1 = |E_k \triangle E_m| \leq \sum_{j \geq k+1} |U_j|. \tag{3.18}$$

This implies that $\mathbb{I}_{E_n}(t)$ converges to a function $f(t)$ in $L^1$. Using Egorov's theorem, we conclude that $f(t) = \mathbb{I}_E(t)$ for a measurable set $E \subset \mathbb{T}$. Tending $m$ to infinity, from (3.17) and (3.18) we get

$$|E_k \triangle E| \leq \left| \bigcup_{j \geq k+1} U_j \right| \leq 2\pi \sum_{j \geq k+1} \delta_j \leq \frac{\gamma\lambda}{16\|\varphi_{r_k}\|_\infty}. \tag{3.19}$$

Fix a point $x \in \mathbb{T}$. We have $x \in J$ where $J$ is an adjacent interval for $E_{k-1}$. From the definition of $E_k$ it follows that either

$$E_k \cap J = J \cap U_k \text{or} E_k \cap J = J \setminus U_k.$$

From (3.15) we have

$$\lambda(r_k, x) = (x - \lambda(r_k), x + \lambda(r_k)) \subset \left[x - \frac{4\pi}{n_k}, x + \frac{4\pi}{n_k}\right].$$

Thus, applying Lemma 3.3, (3.14), and (3.16), we get

$$\begin{aligned}
&\mathrm{OSC}_{\theta \in \lambda(r_k,x)} \Phi_{r_k}(\theta, \mathbb{I}_{E_k}) \geq \mathrm{OSC}_{\theta \in \left[x - \frac{4\pi}{n_k}, x + \frac{4\pi}{n_k}\right]} \Phi_{r_k}(\theta, \mathbb{I}_{E_k}) \\
&\geq \int_{-\frac{\pi\delta_k}{n_k}}^{\frac{\pi\delta_k}{n_k}} \varphi_{r_k}(t)dt - 16\delta_k - 2\pi\varphi_{r_k}\left(\frac{|J|}{4}\right) \\
&\geq \int_{-\delta_k\lambda(r_k)}^{\delta_k\lambda(r_k)} \varphi_{r_k}(t)dt - 16\delta_k - \frac{\gamma\lambda}{8} \\
&\geq \frac{\gamma\lambda}{4} - 16\delta_k.
\end{aligned}$$

From (3.19) we conclude

$$\begin{aligned}
&\mathrm{OSC}_{\theta \in \lambda(r_k,x)} \Phi_{r_k}(\theta, \mathbb{I}_E) \\
&\qquad > \mathrm{OSC}_{\theta \in \lambda(r_k,x)} \Phi_{r_k}(\theta, \mathbb{I}_{E_k}) - \frac{\gamma\lambda}{16} \geq \frac{\gamma\lambda}{8} - 16\delta_k,
\end{aligned}$$

which completes the proof of the theorem since $\delta_k \to 0$. □

## 4 Final Remarks

In the definition of $\lambda(r)$-convergence the range of the parameter $r$ is $(0, 1)$ with the limit point 1. Certainly it is not essential in the theorems. We could take any set $Q \subset \mathbb{R}$ with limit point $r_0$ which is either a finite number or $\infty$. We may define an approximation of the identity on the real line to be a family of functions $\varphi_r \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, $r > 0$, satisfying the same conditions 1)–3) as AI on $\mathbb{T}$ has. We have just made a little change in the condition 2), that is, $\left\|\varphi_r^* \cdot \mathbb{I}_{\{|t|\ge\delta\}}\right\|_1 \to 0$ as $r \to 0$, $\delta > 0$. In this case usually convergence is considered while $r \to 0$. Analogous results can be formulated and proved for the integrals

$$\Phi_r(x, d\mu) = \int_{\mathbb{R}} \varphi_r(t - x) d\mu(t), \quad r > 0. \tag{4.1}$$

And it can be done just repeating the above proofs after little changes.

Any function $\Phi(x) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ defines an approximation of identity by $\varphi_r(x) = r\Phi(x/r)$ as $r \to 0$. The operators corresponding to such kernels in higher-dimensional cases are investigated by E. M. Stein ([18], p. 57). We note for such kernels we have $\|\varphi_r\|_\infty = r^{-1}\|\Phi\|_\infty$ and therefore the condition (2.1) takes the form $\lambda(r) \le cr$. This bound characterizes the nontangential convergence in the upper half-plane and it turns out to be a necessary and sufficient condition for almost everywhere $\lambda(r)$-convergence of the integrals (4.1).

P. Sjögren [15–17], J.-O. Rönning [11–13], I. N. Katkovskaya and V. G. Krotov [6] considered the square root Poisson integrals

$$\mathcal{P}_r^0(x, f) = \frac{1}{c(r)} \int_{\mathbb{T}} [P_r(x - t)]^{1/2} f(t) dt, \tag{4.2}$$

where

$$c(r) = \int_{\mathbb{T}} [P_r(t)]^{1/2} dt$$

is the normalizing coefficient. They proved, that

$$\lim_{r\to 1} \mathcal{P}_r^0(x + \theta(r), f) = f(x) \text{ a.e.}$$

whenever $f \in L^p(\mathbb{T})$, $1 \le p \le \infty$, and

$$|\theta(r)| \le \begin{cases} c(1 - r)\left(\log \frac{1}{1-r}\right)^p & \text{if } 1 \le p < \infty, \\ (1 - r)^{1-\varepsilon}, & \text{if } p = \infty. \end{cases}$$

The case of $p = 1$ is proved in [15], $1 < p \le \infty$ is considered in [11], [12]. They provide also some weak-type inequalities for the maximal operators of square root Poisson integrals. In the paper [6] the authors obtained weighted strong-type inequalities for the same operators. The cases $p = 1$ and $p = \infty$ are consequences of Theorem 2.1 with additional information about the points where the convergence occurs.

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