ON AN EQUIVALENCY OF DIFFERENTIATION BASIS OF DYADIC RECTANGLES

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ABSTRACT. The paper considers differentiation properties of rare basis of dyadic rectangles corresponding to an increasing sequence of integers $\{\nu_k\}$. We prove that the condition

$$\sup_k(\nu_{k+1}-\nu_k)<\infty$$

is necessary and sufficient for such basis to be equivalent to the full basis of dyadic rectangles.

1. INTRODUCTION

Let $\mathcal{R}$ be the family of half-closed rectangles $[a,b)\times[c,d)$ in $\mathbb{R}^2$ and $\mathcal{Q}\subset\mathcal{R}$ be the family of half-closed squares in $\mathbb{R}^2$. Then let $\mathcal{R}^{\text{dyadic}}$ be the family of dyadic rectangles of the form

$$\left[\frac{i-1}{2^n},\frac{i}{2^n}\right)\times\left[\frac{j-1}{2^m},\frac{j}{2^m}\right),\quad i,j,n,m\in\mathbb{Z}, \tag{1}$$

and $\mathcal{Q}^{\text{dyadic}}$ be the family of dyadic squares $(n=m)$. We have $\mathcal{R}^{\text{dyadic}}\subset\mathcal{R}$ and $\mathcal{Q}^{\text{dyadic}}\subset\mathcal{Q}$. For a given rectangle $R\in\mathcal{R}$ we denote by $\operatorname{len}(R)$ the length of the bigger side of R.

Definition 1.1. A family of rectangles $\mathcal{M}\subset\mathcal{R}$ is said to be a differentiation basis (or simply basis), if for any point $x\in\mathbb{R}^2$ there exists a sequence of rectangles $R_k\in\mathcal{M}$ such that $x\in R_k$, $k=1,2,\ldots$ and $\operatorname{len}(R_k)\to 0$ as $k\to\infty$.

Let $\mathcal{M}\subset\mathcal{R}$ be a differentiation basis. For any function $f\in L^1(\mathbb{R}^2)$ we define

$$\delta_{\mathcal{M}}(x,f)=\limsup_{\operatorname{len}(R)\to 0:\, x\in R\in\mathcal{M}}\left|\frac{1}{|R|}\int_R f(t)dt-f(x)\right|.$$

The integral of a function $f\in L^1(\mathbb{R}^2)$ is said to be differentiable at a point $x\in\mathbb{R}^2$ with respect to the basis $\mathcal{M}$, if $\delta_{\mathcal{M}}(x,f)=0$. Consider classes of functions

$$\mathcal{F}(\mathcal{M})=\{f\in L(\mathbb{R}^2):\,\delta_{\mathcal{M}}(x,f)=0\text{ almost everywhere }\},$$
$$\mathcal{F}^+(\mathcal{M})=\{f\in L(\mathbb{R}^2):\,f(x)\ge 0,\,\delta_{\mathcal{M}}(x,f)=0\text{ almost everywhere }\}.$$

Note that $\mathcal{F}(\mathcal{M})$ is the family of functions having almost everywhere differentiable integrals with respect to the basis $\mathcal{M}$.

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Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex function. Denote by $\Phi(L)(\mathbb{R}^2)$ the class of measurable functions f defined on $\mathbb{R}^2$ such that $\Phi(|f|) \in L^1(\mathbb{R}^2)$. If Φ satisfies the Δ_2-condition $\Phi(2x) \le k\Phi(x)$, then $\Phi(L)$ turns to be an Orlicz space with the norm

$$\|f\|_\Phi = \inf\left\{c > 0 : \int_{\mathbb{R}^2} \Phi\left(\frac{|f|}{c}\right) \le 1\right\}.$$

The following classical theorems determine the optimal Orlicz space, which functions have a.e. differentiable integrals with respect to the entire family of rectangles $\mathcal{R}$ is the space

$$L(1 + \log L)(\mathbb{R}^2) \subset L^1(\mathbb{R}^2),$$

corresponding to the case $\Phi(t) = t(1 + \log^+ t)$ ([1]).

Theorem A (Jessen-Marcinkiewicz-Zygmund, [4])**.** $L(1 + \log L)(\mathbb{R}^2) \subset \mathcal{F}(\mathcal{R})$.

Theorem B (Saks, [7])**.** If

$$\Phi(t) = o(t \log t) \text{ as } t \to \infty,$$

then $\Phi(L)(\mathbb{R}^2) \not\subset \mathcal{F}(\mathcal{R})$. Moreover, there exists a positive function $f \in \Phi(L)(\mathbb{R}^2)$ such that $\delta_{\mathcal{R}}(x, f) = \infty$ everywhere.

Such theorems are valid also for the basis $\mathcal{R}^{\text{dyadic}}$. The first one trivially follows from embedding $L(1+\log L)(\mathbb{R}^2) \subset \mathcal{F}(\mathcal{R}) \subset \mathcal{F}(\mathcal{R}^{\text{dyadic}})$. The second can be deduced from the following relation

$$\mathcal{F}^+(\mathcal{R}^{\text{dyadic}}) = \mathcal{F}^+(\mathcal{R}),$$

due to Zerekidze [9] (see also [10, 11]).

Let $\Delta = \{\nu_k : k = 1, 2, \ldots\}$ be an increasing sequence of positive integers. This sequence generates the rare basis $\mathcal{R}_\Delta^{\text{dyadic}}$ of dyadic rectangles of the form (1) with $n, m \in \Delta$. This kind of bases first considered in the papers [8], [2], [3]. Stokolos [8] proved that the analogous of Saks theorem holds for any basis $\mathcal{R}_\Delta^{\text{dyadic}}$ with an arbitrary Δ sequence. That means $L(1+\log L)(\mathbb{R}^2)$ is again the largest Orlicz space containing in $\mathcal{F}(\mathcal{R}_\Delta^{\text{dyadic}})$. G. A. Karagulyan [5] proved some theorems, establishing an equivalency of some convergence conditions for multiple martingale sequences, those in particular imply some results of the papers [8], [2], [3].

In this paper we prove

Theorem. Let $\Delta = \{\nu_k\} \subset \mathbb{N}$ be an increasing sequence of positive integers. Then the condition

$$\sup_{k\in\mathbb{N}}(\nu_{k+1} - \nu_k) < \infty \tag{2}$$

is necessary and sufficient for the equality

$$\mathcal{F}(\mathcal{R}_\Delta^{\text{dyadic}}) = \mathcal{F}(\mathcal{R}^{\text{dyadic}}).$$

2. Some definitions and key functions

Denote by $\overline{E}$ and $\mathring{E}$ the closure and the interior of a set $E \subset \mathbb{R}^2$ respectively, $\mathbb{I}_E$ denotes the indicator function of E. A set $E \subset \mathbb{R}^2$ is said to be simple, if it can be written as a union of squares of the form

$$\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right) \times \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right).$$

If n is the minimal integer with this relation, then we write $\mathrm{wd}(E) = 2^{-n}$. Note that if E is a dyadic rectangle, then $\mathrm{wd}(E)$ coincides with the length of the smaller side of E. If E is square, then $\mathrm{len}(E) = \mathrm{wd}(E)$. Denote

$$E_{ij}(n) = \bigcup_{k=0}^{n-1} \left[\frac{i}{2}, \frac{i}{2} + \frac{1}{2^{k+1}}\right) \times \left[\frac{j}{2}, \frac{j}{2} + \frac{1}{2^{n-k}}\right), \tag{3}$$

$$\begin{aligned} F_{i,j}(n) &= \left[\frac{i}{2}, \frac{i}{2} + \frac{1}{2^n}\right) \times \left[\frac{j}{2}, \frac{j}{2} + \frac{1}{2^n}\right) \\ &= \bigcap_{k=0}^{n-1} \left[\frac{i}{2}, \frac{i}{2} + \frac{1}{2^{k+1}}\right) \times \left[\frac{j}{2}, \frac{j}{2} + \frac{1}{2^{n-k}}\right) \subset E_{ij}(n), \quad i, j = 0, 1, \end{aligned} \tag{4}$$

and define the sets

$$E(n) = E_{00}(n) \cup E_{01}(n) \cup E_{10}(n) \cup E_{11}(n), \tag{5}$$

$$F(n) = F_{00}(n) \cup F_{01}(n) \cup F_{10}(n) \cup F_{11}(n) \subset E(n), \tag{6}$$

Introduce the functions

$$u(x, n) = (n+1)2^{n-2}\big(\mathbb{I}_{F_{00}(n)}(x) + \mathbb{I}_{F_{11}(n)}(x) - \mathbb{I}_{F_{10}(n)}(x) - \mathbb{I}_{F_{01}(n)}\big), \quad n \in \mathbb{N},$$

$$v(x) = \mathbb{I}_{[0,1/2)\times[0,1/2)}(x) + \mathbb{I}_{[1/2,1)\times[1/2,1)}(x) - \mathbb{I}_{[0,1/2)\times[1/2,1)}(x) - \mathbb{I}_{[1/2,1)\times[0,1/2)}(x).$$

Let $\omega \in \mathcal{Q}$ be an arbitrary square and ϕ_ω be the linear transformation of $\mathbb{R}^2$ taking ω onto unit square $[0,1)^2 \subset \mathbb{R}^2$. For an arbitrary function $f(x)$ defined on $[0,1)^2$ and for a set $E \subset [0,1)^2$ we define

$$f_\omega(x) = f(\phi_\omega(x)), \quad E_\omega = (\phi_\omega)^{-1}(E) \subset \omega.$$

We have

$$\mathrm{supp}\,(u_\omega(x, n)) = F_\omega(n), \tag{7}$$

$$\mathrm{supp}\,(v_\omega(x)) = \omega, \tag{8}$$

$$|E_\omega(n)| = \frac{(n+1)|\omega|}{2^n}, \quad |F_\omega(n)| = \frac{|\omega|}{4^{n-1}}, \tag{9}$$

$$\mathrm{wd}\,(E_\omega(n)) = \mathrm{wd}\,(F_\omega(n)) = \mathrm{wd}(\omega) \cdot 2^{-n}. \tag{10}$$

Simple calculations show that

$$\|u_\omega(x, n)\|_1 = |E_\omega(n)| = \frac{n+1}{2^n}|\omega|, \tag{11}$$

$$\|v_\omega(x)\|_1 = |\omega|. \tag{12}$$

Then observe that, if $\omega \in \mathcal{Q}^{\text{dyadic}}$ is a dyadic square, then for any point $x \in E_\omega(n)$ there exists a dyadic rectangle $R(x) \in \mathcal{R}^{\text{dyadic}}$ with

$$\frac{1}{|R(x)|}\left|\int_{R(x)} u_\omega(x,n)dx\right| = \frac{n+1}{2}, \quad x \in R(x) \subset E_\omega(n), \tag{13}$$

$$\operatorname{wd}(R(x)) = \operatorname{wd}(\omega) \cdot 2^{-n}. \tag{14}$$

besides this rectangle coincides with the $(\phi_\omega)^{-1}$-image of one of the representation rectangles from (3) . Similarly, if $\omega \in \mathcal{D}$, then

$$\frac{1}{|R(x)|}\left|\int_{R(x)} v_\omega(x)dx\right| = 1, \quad x \in R(x) \subset \omega, \tag{15}$$

$$\operatorname{wd}(R(x)) = \frac{\operatorname{wd}(\omega)}{2}. \tag{16}$$

for some square $R(x)$ with $|R(x)| = |\omega|/4$. In this case $R(x)$ coincides with one of the four squares forming ω.

3. Auxiliary lemmas

The following simple lemma has been proved in [6].

Lemma 1. Let $Q \in \mathcal{Q}^{\text{dyadic}}$ be an arbitrary dyadic square, a function $f(x) = f(x_1,x_2) \in L^1(\mathbb{R}^2)$ satisfies the condition $\operatorname{supp} f(x) \subset Q$ and

$$\int_{\mathbb{R}} f(x_1,t)dt = \int_{\mathbb{R}} f(t,x_2)dt = 0, \quad x_1,x_2 \in \mathbb{R}. \tag{17}$$

Then for any dyadic rectangle $R \in \mathcal{R}^{\text{dyadic}}$ satisfying $R \not\subset \mathring{Q}$ we have

$$\int_R f(x)dx = 0. \tag{18}$$

Proof. We suppose

$$Q = [\alpha_1,\beta_1) \times [\alpha_2,\beta_2), \quad R = [a_1,b_1) \times [a_2,b_2).$$

If $R \cap Q = \varnothing$, then (18) is trivial. Otherwise we will have either $[\alpha_1,\beta_1) \subset [a_1,b_1)$ or $[\alpha_2,\beta_2) \subset [a_2,b_2)$. In the first case, using (17) , we get

$$\begin{aligned}\int_R f(x)dx &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x_1,x_2)dx_1dx_2 = \int_{a_2}^{b_2}\int_{\alpha_1}^{\beta_1} f(x_1,x_2)dx_1dx_2\\ &= \int_{a_2}^{b_2}\left(\int_{\mathbb{R}} f(x_1,x_2)dx_1\right)dx_2 = 0.\end{aligned}$$

The second case is proved similarly. □

Lemma 2. Let m be a positive integer and Q be a dyadic square. Then for any simple set $E \subsetneq [0,1)^2$, there exists a finite family Ω of dyadic squares

$\omega \subset Q$ such that

$$E_\omega \cap E_{\omega'} = \varnothing, \quad \omega \neq \omega', \tag{19}$$

$$\min_{\omega \in \Omega} \mathrm{wd}(\omega) = \mathrm{wd}(Q) \cdot (\mathrm{wd}(E))^m, \tag{20}$$

$$\left| Q \setminus \bigcup_{\omega \in \Omega} E_\omega \right| = |Q| \left(1 - |E|\right)^m. \tag{21}$$

Proof. Define a sequence of sets G_k, $k = 1, 2, \ldots, m$, with

$$Q = G_1 \supset G_2 \supset \ldots \supset G_m, \tag{22}$$

and finite families of dyadic squares $\Omega_k \subset \mathcal{D}$, $k = 1, 2, \ldots, m+1$, such that

$$\mathrm{wd}(\omega) = \mathrm{wd}(Q) \cdot (\mathrm{wd}(E))^{k-1}, \quad \omega \in \Omega_k, \quad k = 1, 2, \ldots, m+1, \tag{23}$$

$$G_k = \bigcup_{\omega \in \Omega_k} \omega, \quad k = 1, 2, \ldots, m+1, \tag{24}$$

$$G_k = G_{k-1} \setminus \bigcup_{\omega \in \Omega_{k-1}} E_\omega = \bigcup_{\omega \in \Omega_{k-1}} \left(\omega \setminus E_\omega\right), \quad k = 2, \ldots, m+1. \tag{25}$$

We do it by induction. For the first step of induction we take just $G_1 = Q$ and let Ω_1 consist of a single rectangle Q. Suppose we have already chosen the sets G_k and the families Ω_k for $k = 1, 2, \ldots, p$, satisfying (22) -(25) . Set

$$G_{p+1} = G_p \setminus \bigcup_{\omega \in \Omega_p} E_\omega = \bigcup_{\omega \in \Omega_p} \left(\omega \setminus E_\omega\right).$$

From the induction hypothesis of (23) it follows that

$$\mathrm{wd}\left(\omega \setminus E_\omega\right) = \mathrm{wd}(\omega) \cdot \mathrm{wd}(E) = \mathrm{wd}(Q) \cdot (\mathrm{wd}(E))^p.$$

Hence we conclude that G_{p+1} is a union of dyadic squares with side lengths $\mathrm{wd}(Q) \cdot (\mathrm{wd}(E))^p$ and we define the family Ω_{p+1} as a collection of these squares. Thus we get G_{p+1} and Ω_{p+1} satisfying the conditions (22) -(25) for $k = p+1$, that completes the induction process. Applying (11) , (24) and (25) we obtain

$$|G_k| = |G_{k-1}| - \left| \bigcup_{\omega \in \Omega_{k-1}} E_\omega \right| = |G_{k-1}| - |E||G_{k-1}| = \left(1 - |E|\right)|G_{k-1}|$$

and therefore

$$|G_{m+1}| = \left(1 - |E|\right)^m |Q|. \tag{26}$$

Obviously the family of squares $\Omega = \cup_{k=1}^{m+1} \Omega_k$ satisfies the hypothesis of lemma. Indeed, suppose $\omega, \omega' \in \Omega$ are arbitrary squares. If $\omega, \omega' \in \Omega_k$ for some k, then according to (23) we have $\omega \cap \omega' = \varnothing$ and so (19) . If $\omega \in \Omega_k$, $\omega' \in \Omega_{k'}$ and $k < k'$, then

$$E_{\omega'} \subset \omega' \subset G_{k'},$$
$$E_\omega \subset G_k \setminus G_{k+1} \Rightarrow E_\omega \cap G_{k'} = \varnothing.$$

Thus we again get (19) . The condition (20) immediately follows from (23) , and (21) follows from (26) and from the relation

$$\left|\bigcup_{\omega\in\Omega} E_\omega\right| = \left|\bigcup_{k=1}^{m+1}\bigcup_{\omega\in\Omega_k} E_\omega\right| = \left|\bigcup_{k=1}^{m+1} G_k\setminus G_{k+1}\right| = |Q\setminus G_{m+1}| = |Q|(1-(1-|E|)^m).$$

□

Lemma 3. Let $L>1$ be a positive integer and let $Q\in\mathcal{D}$ be a dyadic square. Then there exist a function $f\in L^\infty(\mathbb{R}^2)$ and numbers $\alpha(L)\in\mathbb{N},\beta(L)>0$, depended on L such that

$$\operatorname{supp} f\subset Q, \tag{27}$$

$$\|f\|_\infty\le\beta(L), \tag{28}$$

$$|\operatorname{supp} f|\le\frac{2|Q|}{\beta(L)}, \tag{29}$$

$$\operatorname{wd}(\operatorname{supp} f)\ge\operatorname{wd}(Q)\cdot 2^{-\alpha(L)}, \tag{30}$$

$$\int_R f(x)dx=0,\quad R\in\mathcal{R}^{\text{dyadic}},\quad R\not\subset\mathring{Q}, \tag{31}$$

and for any point $x\in Q$ there exists a rectangle $R(x)\subset Q$ satisfying

$$\operatorname{wd}(R(x))\ge\operatorname{wd}(Q)\cdot 2^{-\alpha(L)}, \tag{32}$$

$$\frac{1}{|R(x)|}\left|\int_{R(x)} f(t)dt\right|\ge L. \tag{33}$$

Proof. Let $n=2L$ and denote

$$\alpha(L)=n(2^n+1),\quad \beta(L)=(n+1)2^{n-2}, \tag{34}$$

$$m=m(L)=\left[\frac{2^n(\ln(n+1)+(n-2)\ln 2)}{n+1}\right]+1<2^n. \tag{35}$$

Let $E=E(n)$ be the set defined in (5) . We have $|E(n)|=(n+1)/2^n$ and $\operatorname{wd}(E(n))=2^{-n}$. Applying Lemma 2, we may find family Ω of dyadic squares $\omega\subset Q$ with properties (19) -(21) . Set

$$G=\bigcup_{\omega\in\Omega} E_\omega(n),\quad G_1=Q\setminus G. \tag{36}$$

According to (21) , (34) and (35) , we have

$$|G_1|=(1-|E(n)|)^m\,|Q|=\left(1-\frac{n+1}{2^n}\right)^m|Q|<\frac{|Q|}{\beta(L)}$$

From (20) and (35) it follows that

$$G_1=\bigcup_{\omega\in\Omega_1}\omega,$$

where Ω_1 is a family of squares with

$$\min_{\omega\in\Omega_1}\operatorname{wd}(\omega)=\min_{\omega\in\Omega}\operatorname{wd}(\omega)=\operatorname{wd}(Q)\cdot(\operatorname{wd}(E(n)))^m\ge\operatorname{wd}(Q)\cdot 2^{-n\cdot 2^n}. \tag{37}$$

Define

$$f(x)=\sum_{\omega\in\Omega}u_\omega(x,n)+\beta(L)\sum_{\omega\in\Omega_1}v_\omega(x)=g(x)+g_1(x).$$

Clearly this function satisfies (27) and (28). Then, we have

$$\operatorname{supp} g=\bigcup_{\omega\in\Omega}F_\omega(n)\subset G,\quad \operatorname{supp} g_1=G_1,$$
$$\operatorname{supp} f=\operatorname{supp} g\bigcup\operatorname{supp} g_1.$$

This together with (9) and (36) implies

$$|\operatorname{supp} f|=\bigcup_{\omega\in\Omega}|F_\omega(n)|+|G_1|=\frac{1}{(n+1)2^{n-2}}\sum_{\omega\in\Omega}|E_\omega(n)|+|G_1|$$
$$=\frac{1}{(n+1)2^{n-2}}|G|+|G_1|\le\frac{2|Q|}{\beta(L)}$$

and therefore we get (29). Using (37), we obtain

$$\operatorname{wd}(\operatorname{supp} g)\ge\min_{\omega\in\Omega}\operatorname{wd}(\omega)\cdot\operatorname{wd}(F(n))=\operatorname{wd}(Q)\cdot 2^{-n(2^n+1)}=\operatorname{wd}(Q)\cdot 2^{-\alpha(L)},$$
$$\operatorname{wd}(\operatorname{supp} g_1)\ge\min_{\omega\in\Omega_1}\operatorname{wd}(\omega)\ge\operatorname{wd}(Q)\cdot 2^{-n\cdot 2^n}>\operatorname{wd}(Q)\cdot 2^{-\alpha(L)},$$

and therefore we get (30). The condition (31) follows from Lemma 1, since $f(x)$ satisfies the condition (17) according the definitions of functions $u_\omega(x,n)$ and $v_\omega(x)$. To prove (33) we take an arbitrary point $x\in Q$. We have either $x\in G$ or $x\in G_1$. In the first case we will have $x\in E_\omega(n)$ for some square $\omega\in\Omega$. By (13) there exists a dyadic rectangle $R=R(x)$, $x\in R\subset E_\omega(n)$, such that

$$\frac{1}{|R|}\left|\int_R f(t)dt\right|=\frac{1}{|R|}\left|\int_R u_\omega(t,n)dt\right|=\frac{n+1}{2}>L.$$

In the second case from (15) we obtain

$$\frac{1}{|R|}\left|\int_R f(t)dt\right|=\frac{\beta(L)}{|R|}\left|\int_R v_\omega(t)dt\right|\ge 2^n>L$$

for some square $R=R(x)$, $x\in R\subset\omega$. Obviously in any case $R(x)$ satisfies (32). Lemma is proved. □

Proof of Theorem. Necessity : Let $\Delta=\{\nu_k\}$ be a sequence with

$$\gamma=\sup_{k\in\mathbb{N}}(\nu_{k+1}-\nu_k)<\infty, \tag{38}$$

and suppose conversely, we have

$$\mathcal{F}(\mathcal{R}_\Delta^{\text{dyadic}})\setminus\mathcal{F}(\mathcal{R}^{\text{dyadic}})\ne\varnothing.$$

That means there exists a function $f\in L^1(\mathbb{R}^2)$ such that

$$\delta_{\mathcal{R}_\Delta^{\text{dyadic}}}(x,f)=0\text{ a.e.}, \tag{39}$$

$$\delta_{\mathcal{R}^{\text{dyadic}}}(x,f)>\alpha,\quad x\in E, \tag{40}$$

where $\alpha > 0$ and $|E| > 0$. According to (39) for any $x \in \mathbb{R}^2$ one can chose a number $\delta(x) > 0$ such that the conditions

$$x \in R \in \mathcal{R}_\Delta^{\text{dyadic}}, \quad \text{len}(R) < \delta(x),$$

imply

$$\left|\frac{1}{|R|}\int_R f - f(x)\right| < \frac{\alpha}{2}. \tag{41}$$

For some $\delta > 0$ the set $F = \{x \in E\colon \delta(x) \geq \delta\} \subset E$ has positive measure. Then, using the representation

$$F = \bigcup_{j\in\mathbb{Z}} \left\{x \in F\colon \frac{j\alpha}{2} \leq f(x) < \frac{(j+1)\alpha}{2}\right\},$$

we find a set

$$G = \left\{x \in F\colon \frac{j_0\alpha}{2} \leq f(x) < \frac{(j_0+1)\alpha}{2}\right\} \subset F \tag{42}$$

having positive measure. Combining (40), (41) and (42), we will have

$$\delta_{\mathcal{R}^{\text{dyadic}}}(x, f) > \alpha, \quad x \in G, \tag{43}$$

$$\left|\frac{1}{|R|}\int_R f - f(x)\right| < \frac{\alpha}{2}, \text{ if } x \in R \cap G,\ R \in \mathcal{R}_\Delta^{\text{dyadic}},\ \text{len}(R) < \delta, \tag{44}$$

$$\sup_{x,y\in G} |f(x) - f(y)| \leq \frac{\alpha}{2}. \tag{45}$$

Since almost all points of G are density points, we may fix $x_0 \in G$ with

$$\lim_{\text{len}(R)\to 0,\, x_0\in R\in\mathcal{R}^{\text{dyadic}}} \frac{|R\cap G|}{|R|} = 1.$$

Using this relation and (43), we find a rectangle

$$R' = \left[\frac{p-1}{2^n}, \frac{p}{2^n}\right) \times \left[\frac{q-1}{2^m}, \frac{q}{2^m}\right),$$

such that

$$x_0 \in R' \in \mathcal{R}^{\text{dyadic}}, \quad \text{len}(R') < \delta, \tag{46}$$

$$\left|\frac{1}{|R'|}\int_{R'} f - f(x_0)\right| > \alpha, \tag{47}$$

$$|R' \cap G| > (1 - 4^{-\gamma})|R'|, \tag{48}$$

where γ is the number (38). Besides, we may suppose

$$\nu_{k_t-1} < n \leq \nu_{k_t}, \quad \nu_{k_s-1} < m \leq \nu_{k_s}, \tag{49}$$

for some integers t, s. This and (38) imply that R' is a union of rectangles of the form

$$\left[\frac{i-1}{2^{\nu_{k_t}}}, \frac{i}{2^{\nu_{k_t}}}\right) \times \left[\frac{j-1}{2^{\nu_{k_s}}}, \frac{j}{2^{\nu_{k_s}}}\right) \in \mathcal{R}_\Delta^{\text{dyadic}},$$

and from (47) it follows that at least for one of these rectangles, say R'', we have

$$\left|\frac{1}{|R''|}\int_{R''} f - f(x_0)\right| > \alpha. \tag{50}$$

From (38) and (49) we get

$$|R''| = \frac{1}{2^{\nu_{k_t}+\nu_{k_s}}} \ge \frac{1}{2^{\nu_{k_t}+\nu_{k_s}-\nu_{k_t-1}-\nu_{k_s-1}}}\cdot\frac{1}{2^{n+m}} \ge |R'|\cdot 4^{-\gamma}.$$

From this and (48) we obtain $R''\cap G \ne \varnothing$. Take a point $x_1 \in R''\cap G$. From (45) and (50) we get

$$\left|\frac{1}{|R''|}\int_{R''} f - f(x_1)\right| > \left|\frac{1}{|R''|}\int_{R''} f - f(x_0)\right| - |f(x_1)-f(x_0)| > \frac{\alpha}{2}. \tag{51}$$

On the other hand we have $x_1 \in R''\cap G$, $R'' \in \mathcal{R}_\Delta^{\mathrm{dyadic}}$, $\mathrm{len}(R'') \le \mathrm{len}(R') < \delta_0$, and therefore by (44) we obtain

$$\left|\frac{1}{|R''|}\int_{R''} f - f(x_1)\right| < \alpha/2.$$

The last relation together with (51) gives a contradiction, which completes the proof of the first part of our theorem.

Sufficiency: Now we suppose (2) doesn't hold, that means there exists a sequence of integers $p_k \nearrow \infty$ such that

$$\lim_{k\to\infty}(\nu_{p_k+1}-\nu_{p_k}) = \infty. \tag{52}$$

Using this relation, we may find sequences of integers L_k and l_k, $k = 1,2,\ldots$, such that

$$l_{k+1} > l_k + \alpha(L_k), \quad k = 1,2,\ldots, \tag{53}$$

$$\nu_{p_k} < l_k < l_k + \alpha(L_k) < \nu_{p_k+1}, \quad k = 1,2,\ldots, \tag{54}$$

$$L_{k+1} > 2^k\cdot(\beta(L_k)+k) \quad k = 1,2,\ldots, \tag{55}$$

where $\alpha(L)$ and $\beta(L)$ are the constants taken from Lemma 3. Applying Lemma 3 for the numbers $L = L_k$, $l = l_k$ and for the square

$$Q = Q_{ij}^k = \left[\frac{i-1}{2^{l_k}}, \frac{i}{2^{l_k}}\right) \times \left[\frac{j-1}{2^{l_k}}, \frac{j}{2^{l_k}}\right), \quad 1 \le i,j \le 2^{l_k},$$

we get functions $f_{ij}^k(x) \in L^\infty(\mathbb{R}^2)$ satisfying the conditions

$$\operatorname{supp} f_{ij}^k \subset Q_{ij}^k, \tag{56}$$

$$\|f_{ij}^k\|_\infty \le \beta(L_k), \tag{57}$$

$$|\operatorname{supp} f_{ij}^k| \le \frac{2|Q_{ij}^k|}{\beta(L_k)}, \tag{58}$$

$$\mathrm{wd}(\operatorname{supp} f_{ij}^k) \ge 2^{-l_k-\alpha(L_k)}, \tag{59}$$

$$\int_R f_{ij}^k(x)dx = 0, \quad R \in \mathcal{R}^{\mathrm{dyadic}}, \quad R \not\subset \mathring{Q}_{ij}^k, \tag{60}$$

and for any point $x \in Q_{ij}^k$ there exists a dyadic rectangle $R_k(x) \subset Q_{ij}^k$ with

$$\operatorname{wd}(R_k(x)) \geq 2^{-l_k-\alpha(L_k)}, \tag{61}$$

$$\frac{1}{|R_k(x)|}\left|\int_{R_k(x)} f_{ij}^k(t)dt\right| \geq L_k. \tag{62}$$

Define the function

$$F_k(x) = \sum_{i,j=1}^{2^{l_k}} f_{ij}^k(x).$$

From the relations (56)-(62) we conclude

$$|\operatorname{supp} F_k| \leq \frac{2}{\beta(L_k)}, \tag{63}$$

$$\operatorname{wd}(\operatorname{supp} F_k) \geq 2^{-l_k-\alpha(L_k)}, \tag{64}$$

$$\|F_k\|_\infty \leq \beta(L_k), \tag{65}$$

$$\int_R F_k(x)dx = 0, \quad R \in \mathcal{R}^{\text{dyadic}}, \quad \operatorname{len}(R) \geq 2^{-l_k}, \tag{66}$$

and for any point $x \in [0,1)^2$ there exists a dyadic rectangle $R_k(x) \subset [0,1)^2$ such that

$$2^{-l_k} > \operatorname{len}(R_k(x)) \geq \operatorname{wd}(R_k(x)) \geq 2^{-l_k-\alpha(L_k)}, \tag{67}$$

$$\frac{1}{|R_k(x)|}\left|\int_{R_k(x)} F_k(t)dt\right| \geq L_k. \tag{68}$$

Denote

$$F(x) = \sum_{k=1}^{\infty} \frac{F_k(x)}{2^k}. \tag{69}$$

From (63) and (54) it follows that $\|F_k\|_1 \leq 2$ and so $\|F\|_1 \leq 2$. Let $x \in [0,1)^2$ be an arbitrary point. From the relations (53) and (67) we get $\operatorname{len}(R_k(x)) \geq 2^{-l_{k+1}} \geq 2^{-l_j}$ if $j > k$. Thus, using (66), we obtain

$$\int_{R_k(x)} F_j(t)dt = 0, \quad j > k. \tag{70}$$

On the other hand the relations (65) and (55) imply

$$\left|\frac{1}{|R_k(x)|}\int_{R_k(x)} \sum_{j=1}^{k-1} \frac{F_j(t)}{2^j}dt\right| \leq \beta(L_{k-1}) < \frac{L_k}{2}, \quad k \geq 2. \tag{71}$$

From (68), (70) and (71) we get the inequality

$$\left|\frac{1}{|R_k(x)|}\int_{R_k(x)} F(t)dt\right| \geq \frac{1}{|R_k(x)|}\left|\int_{R_k(x)} F_k(t)dt\right| - \frac{L_k}{2} > \frac{L_k}{2},$$

which yields

$$\limsup_{\operatorname{len}(R)\to 0,\, x\in R\in\mathcal{R}^{\text{dyadic}}} \left|\frac{1}{|R|}\int_R F(t)dt\right| = \infty, \quad x \in [0,1)^2. \tag{72}$$

Now take an arbitrary rectangle $R \in \mathcal{R}_\Delta^{\text{dyadic}}$. We have

$$\operatorname{len}(R) = 2^{-\nu_k} \geq \operatorname{wd}(R) = 2^{-\nu_t}. \tag{73}$$

From (66) we get

$$\int_R F_j(t)dt = 0 \text{ if } l_j \geq \nu_k. \tag{74}$$

On the other hand if $l_j < \nu_k$, then from (54) it follows that

$$l_j + \alpha(L_j) < \nu_k$$

and therefore by (64) we get

$$\operatorname{wd}(\operatorname{supp}(F_j)) \geq 2^{-l_j-\alpha(L_j)} \geq 2^{-\nu_k}. \tag{75}$$

Thus, using simple properties of dyadic rectangles, we conclude that

$$l_j < \nu_k,\ R \not\subset \operatorname{supp}(F_j) \Rightarrow R \cap \operatorname{supp}(F_j) = \varnothing. \tag{76}$$

Consider the sets

$$\begin{aligned}
&G_1 = \{x \in [0,1)^2 :\ \delta_{\mathcal{R}}(x, F_k) = 0,\ k = 1, 2, \ldots\},\\
&G_2 = \bigcup_{k=1}^{\infty} \bigcap_{j:\, l_j \geq \nu_k}^{\infty} \Big([0,1)^2 \setminus \operatorname{supp}(F_j) \Big),\\
&G = G_1 \cap G_2.
\end{aligned}$$

Since $F_k(x)$ is bounded, the equality $\delta_{\mathcal{R}}(x, F_k) = 0$ holds almost everywhere and so $|G_1| = 1$. From (63) it follows that $|G_2| = 1$ and therefore we get $|G| = 1$. Take an arbitrary point $x \in G$. We have

$$x \notin \operatorname{supp}(F_j), \quad j > k_0, \tag{77}$$

for some k_0. Consider the rectangle $R \in \mathcal{R}_\Delta^{\text{dyadic}}$ such that $x \in R$. Suppose we have (73) and $k > k_0$. Then form (76) and (77) we get

$$R \cap \operatorname{supp}(F_j) = \varnothing, \text{ if } j > k_0 \text{ and } l_j < \nu_k. \tag{78}$$

From (74) and (78) we conclude

$$\frac{1}{|R|} \int_R F(t)dt = \sum_{j=1}^{k_0} \frac{1}{2^j \cdot |R|} \int_R F_j(t)dt.$$

Thus we obtain

$$\lim_{\operatorname{len}(R)\to 0,\, x \in R \in \mathcal{R}_\Delta^{\text{dyadic}}} \frac{1}{|R|} \int_R F(t)dt = \sum_{j=1}^{k_0} \frac{F_j(x)}{2^j}. \tag{79}$$

On the other hand (77) implies

$$F(x) = \sum_{j=1}^{k_0} \frac{F_j(x)}{2^j}. \tag{80}$$

From (72), (79) and (80) we conclude the relation $F \in \mathcal{F}(\mathcal{R}_\Delta^{\text{dyadic}}) \setminus \mathcal{F}(\mathcal{R}^{\text{dyadic}})$, which completes the proof of the theorem. $\square$

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