ON CLASSES OF EVERYWHERE DIVERGENT POWER SERIES

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Abstract. We prove the everywhere divergence of series

$$\sum_{n=0}^{\infty} a_n e^{i\rho_n} e^{inx},$$

and

$$\sum_{n=0}^{\infty} (-1)^{[\rho_n]} a_n \cos nx,$$

for sequences a_n and ρ_n satisfying some extremal conditions. These results generalize some well known examples of everywhere divergent power and trigonometric series.

1. Introduction

We consider power series

$$\sum_{n=0}^{\infty} c_n e^{inx} \tag{1}$$

with complex coefficients c_n. In 1912 Luzin [8] constructed an example of everywhere divergent series (1) with $c_n \to 0$. It was an answer to Fatou's problem [4], asking whether power series converge almost everywhere, if $c_n \to 0$. Then in 1912 Steinhaus [11] proved that the trigonometric series

$$\sum_{n=2}^{\infty} \frac{\cos n(x + \log\log n)}{\log n} \tag{2}$$

is everywhere divergent. In 1916 Hardy and Littlewood [6] proved that the power series

$$\sum_{n=1}^{\infty} n^{-\rho} e^{i\alpha n \log n} e^{inx}, \quad 0 < \rho \le \frac{1}{2}, \quad \alpha > 0. \tag{3}$$

1991 *Mathematics Subject Classification.* 42B25.

Key words and phrases. everywhere divergent series, power series, trigonometric series.

diverges for any $x \in \mathbb{R}$. Many authors were interested in the problem of possible upper bound for the coefficients of everywhere divergent power series. In 1921 Neder [9] proved that if

$$a_n \downarrow 0, \quad \sum_{n=1}^{\infty} a_n^2 = \infty, \tag{4}$$

then there exists a power series (1), which is everywhere divergent and $c_n = O(a_n)$. Note that the second condition in (4) is necessary for almost everywhere divergence of series (1) according to Carleson's well known theorem ([2]). Generalizing Neder's theorem Stechkin [10] constructed an example of series (1), which real an imaginary parts are everywhere divergent and $c_n = O(a_n)$, where a_n satisfies (4). Then Herzog constructed an example of series

$$\sum_{n=1}^{\infty} a_n \cos nx$$

with $a_n \geq 0$, which is everywhere divergent. Dvoretzky and Erdos [3] proved that if

$$|a_n| \geq |a_{n+1}|, \quad \sum_{n=1}^{\infty} |a_n|^2 = \infty,$$

then there exists a sequence of numbers $\varepsilon_n = 0$ or 1, such that the series

$$\sum_{n=1}^{\infty} \varepsilon_n a_n e^{inx}$$

is everywhere divergent.

In 1985 Galstyan [5] proved that if a sequence a_n satisfies the conditions (4), then there exists a sequence $\varepsilon_n = 0$ or 1, such that the series

$$\sum_{n=1}^{\infty} \varepsilon_n a_n \cos nx$$

diverges for any $x \in \mathbb{R}$. It was the solution of a problem posed by Ul'yanov [12] in 1964.

We denote by

$$\Delta\, x_n = x_{n+1} - x_n, \quad \Delta^2 x_n = \Delta\, x_{n+1} - \Delta\, x_n.$$

the first and second order differences of a given sequence of numbers x_n, $n = 1, 2, \ldots$.

Theorem 1. *If a sequence ρ_n, $n = 1, 2, \ldots$, satisfies the conditions*

$$\Delta^2 \rho_n \downarrow 0, \quad \sum_{n=1}^{\infty} \Delta^2 \rho_n = \infty, \tag{5}$$

and

$$a_n \geq c \cdot \sqrt{\Delta^2 \rho_n}, \quad n = 1, 2, \ldots, \quad c > 0, \tag{6}$$

then the real and imaginary parts of the series

$$\sum_{n=0}^{\infty} a_n e^{i\rho_n} e^{inx}, \tag{7}$$

diverge for any $x \in \mathbb{R}$.

Theorem 2. *If ρ_n, $n = 1, 2, \ldots$, satisfies the conditions (5) and (6) and $a_n \downarrow 0$, then the series*

$$\sum_{n=0}^{\infty} \varepsilon_n a_n \cos nx, \quad \varepsilon_n = \frac{(-1)^{[\rho_n]} + 1}{2}, \tag{8}$$

diverges for any $x \in \mathbb{R}$.

Remark 1. *For a given number $\alpha \in \mathbb{R}$ we shall use notation*

$$\alpha/2\pi = 2\pi \left\{ \frac{\alpha}{2\pi} \right\},$$

where $\{x\}$ denotes the fractional part of x. Note that in (5) and (6) we may everywhere replace $\Delta^2 \rho_n$ with $(\Delta^2 \rho_n)/2\pi$.

Remark 2. *Notice that the values of ε_n in (8) are 0 or 1. Without a significant change in the proof of Theorem 2 one can deduce also the everywhere divergence of series*

$$\sum_{n=0}^{\infty} (-1)^{[\rho_n]} a_n \cos nx.$$

Remark 3. *For given coefficients a_n, satisfying (4), we consider the sequence*

$$\rho_1 = 0, \quad \rho_n = \sum_{k=0}^{n-2} (n - k - 1) a_k^2, \quad n = 2, 3, \ldots. \tag{9}$$

It is easy to check that

$$\Delta^2 \rho_n = a_n^2, \quad n = 1, 2, \ldots$$

and hence we have (5) and (6). Thus we conclude that for a given sequence of coefficients (4), defining ρ_n by (9), we get series (7) and (8) diverging at any point $x \in \mathbb{R}$. So the theorems provide concrete *examples of everywhere divergence trigonometric and power series with coefficients having extremal rate. Moreover, the examples of series (2), (3) and the series constructed in the papers* [9, 7, 10, 5] *may be immediately deduced from the Theorems 1 or 2.*

2. Auxiliary lemmas

Lemma 1. *If $b_1 \geq b_2 \geq \ldots \geq b_m \geq 0$, then*

$$\left(\sum_{j=1}^{n} b_j\right)^2 \geq \sum_{j=1}^{m} j b_j^2, \quad n \leq m \leq n\sqrt{2} - 1. \tag{10}$$

Proof. Observe that

$$b_i \sum_{j=i+1}^{n} b_j \geq \sum_{j=i+1}^{n} b_j^2.$$

Thus, we get

$$\sum_{1 \leq i < j \leq n} b_i b_j = \sum_{i=1}^{n-1} \left(b_i \sum_{j=i+1}^{n} b_j \right) \geq \sum_{j=2}^{n} (j-1) b_j^2$$

and therefore

$$\sum_{j=1}^{n} b_j^2 + \sum_{1 \leq i < j \leq n} b_i b_j \geq \sum_{j=1}^{n} j b_j^2. \tag{11}$$

On the other hand we have

$$\begin{aligned}
&\sum_{1 \leq i < j \leq n} b_i b_j \\
&\qquad \geq \frac{n(n-1)}{2} b_n^2 \\
&\qquad \geq \left(\frac{n\sqrt{2}(n\sqrt{2}-1)}{2} - \frac{n(n+1)}{2} \right) b_n^2 \\
&\qquad \geq \left(\frac{m(m+1)}{2} - \frac{n(n+1)}{2} \right) b_n^2 \\
&\qquad = (n+1+n+2+\ldots+m) b_n^2 \\
&\qquad \geq \sum_{j=n+1}^{m} j b_j^2.
\end{aligned} \tag{12}$$

Combining the formula

$$\left(\sum_{j=1}^{n} b_j\right)^2 = \sum_{j=1}^{n} b_j^2 + 2\sum_{1\le i<j\le n} b_i b_j. \tag{13}$$

with the inequalities (11) and (12), we get (10). □

Lemma 2. *If* $b_1 \ge b_2 \ge \ldots \ge b_m \ge 0$, $15 \le n \le m$, *and*

$$c_1 = \sum_{j=1}^{m} j b_j^2, \quad c_2 = \left(\sum_{j=n+1}^{m} (j-n) b_j^2\right) \tag{14}$$

then

$$\sum_{j=1}^{n} j b_j^2 \ge \frac{(c_1-c_2)^2}{9c_1}. \tag{15}$$

Proof. Denote

$$c = \sum_{j=1}^{n} j b_j^2. \tag{16}$$

We have

$$b_n^2 = \frac{2b_n^2}{n(n+1)} \sum_{j=1}^{n} j \le \frac{2}{n(n+1)} \sum_{j=1}^{n} j b_j^2 \le \frac{2c}{n^2}. \tag{17}$$

If $c < c_1 \le 2c$, then (15) is trivial. So we may suppose $c_1 > 2c$. Since $n \ge 15$ there exists an integer p such that

$$n \le n\sqrt{\frac{c_1}{2c}} \le p \le \frac{11n}{10}\sqrt{\frac{c_1}{2c}}. \tag{18}$$

Using the relations (14), (16), (17) and (18), we obtain

$$\begin{aligned} c_1 - c_2 &= c + n \sum_{j=n+1}^{m} b_j^2 \\ &\le c + \frac{2c(p-n)}{n} + n \sum_{j=p+1}^{m} b_j^2 \\ &\le \frac{2cp}{n} + \frac{nc_1}{p} \le \frac{11}{10}\sqrt{2c_1 c} + \sqrt{2c_1 c} < 3\sqrt{c_1 c}, \end{aligned} \tag{19}$$

where we assume

$$\sum_{j=p+1}^{m} b_j^2 = 0$$

if $p \ge m$. After simple calculations from (19) we get (15). □

Lemma 3. *If a sequence x_n satisfies the conditions (5) and $l \geq 0$ is an arbitrary integer, then there exist integers p and q, $q > p > l$, such that*

$$0 < (x_j)/2\pi < \frac{\pi}{50}, \quad p \leq j \leq q, \tag{20}$$

$$\sum_{j=p}^{q} \sqrt{\Delta^2 x_j} > 10^{-3}. \tag{21}$$

Proof. Define the sequence of integers n_k satisfying

$$\Delta x_{n_k} < 2\pi k \leq \Delta x_{n_k+1}. \tag{22}$$

Since $\Delta^2 x_k \to 0$, there exists a number k_0, such that

$$\Delta^2 x_{n_{k-1}} < \pi \cdot 10^{-7}, \quad n_{k-1} > l, \quad k > k_0. \tag{23}$$

For a fixed integer $k > k_0$ we consider the sequence

$$\gamma_n = \sum_{j=n}^{n_k-1} (j-n+1)\Delta^2 x_j, \quad n_{k-1} < n < n_k. \tag{24}$$

We also assume $\gamma_{n_k} = 0$. Observe that, if

$$\gamma_n < 3\pi, \quad n_{k-1} < n < n_k - 1000, \tag{25}$$

then

(26)

$$\begin{aligned}\gamma_{n-1} - \gamma_n = \sum_{j=n-1}^{n_k-1} \Delta^2 x_j \\ &\leq 1001 \cdot \Delta^2 x_{n_{k-1}} + \frac{1}{1000} \sum_{j=n+999}^{n_k-1} (j-n+1)\Delta^2 x_j \\ &\leq 1001 \cdot \pi \cdot 10^{-7} + \gamma_n \cdot 10^{-3} < \frac{\pi}{300}.\end{aligned}$$

From (24) and (25) we also get

$$\Delta^2 x_{n_k} \leq \frac{2\gamma_n}{(n_k-n)^2} \leq \frac{6\pi}{1000(n_k-n)} < \frac{\pi}{100(n_k-n)}. \tag{27}$$

Define the integers p, q, $n_{k-1} < p < q < n_k$, satisfying

$$\begin{aligned}\gamma_{q+1} + \alpha < 2\pi \quad &\leq \gamma_q + \alpha, \\ \gamma_p + \alpha \leq 2\pi + \frac{\pi}{100} \quad &< \gamma_{p-1} + \alpha.\end{aligned} \tag{28}$$

where

$$\alpha = (x_{n_k})/2\pi. \tag{29}$$

From (26) and (28) we obtain

$$\gamma_q \leq \gamma_{q+1} + \frac{\pi}{300} < 2\pi - \alpha + \frac{\pi}{300},$$
$$\gamma_p \geq \gamma_{p-1} - \frac{\pi}{300} \geq 2\pi - \alpha + \frac{\pi}{100} - \frac{\pi}{300} = 2\pi - \alpha + \frac{2\pi}{300}.$$

Using these estimates and the inequalities (10) and (15), we get

$$\left(\sum_{j=p}^{q} \sqrt{\Delta^2 x_j}\right)^2 > \frac{(\gamma_p - \gamma_q)^2}{9\gamma_q} > 10^{-6}. \tag{30}$$

It is easy to verify, that for any numbers $n < m$ we have

$$\begin{aligned} x_n - x_{n_k} &= -\sum_{j=n}^{n_k-1} \Delta x_j \\ &= \sum_{j=n}^{n_k-1} (j-n+1)\Delta^2 x_j - (n_k - n)\Delta x_{n_k} \\ &= \gamma_n - (n_k - n)\Delta x_{n_k}. \end{aligned} \tag{31}$$

Since $\Delta x_{n_k+1} - \Delta x_{n_k} = \Delta^2 x_{n_k}$, from (22) it follows that

$$\Delta x_{n_k} = 2\pi k - \theta \Delta^2 x_{n_k}, \quad 0 \leq \theta \leq 1. \tag{32}$$

Substituting this in (31), we obtain

$$x_n - x_{n_k} = \gamma_n + \theta(n_k - n)\Delta^2 x_{n_k} - 2\pi k(n_k - n),$$

which implies

$$x_n = \alpha + \gamma_n + \theta(n_k - n)\Delta^2 x_{n_k} \mod 2\pi, \tag{33}$$

where α is defined in (29). From (27), (33) and (28) it follows that

$$(x_n)/2\pi \in [0, \pi/50], \quad p \leq n \leq q,$$

and this completes the proof. □

Lemma 4. *If*

$$x \neq \frac{\pi}{2} \mod 2\pi, \tag{34}$$

then there exists an integer $\nu = \nu(x) > 10$, such that

$$\#\{k \in \mathbb{N} : n < k \leq n + \nu, \quad |\cos kx| \geq \sin(\pi/20)\} \geq \frac{3\nu}{5} \tag{35}$$

for any $n = 1, 2, \ldots$.

Proof. If $x/2\pi = 0$ (35) is immediate. Consider the case if $x/2\pi$ is a rational number and

$$x = \frac{2\pi q}{p} \mod 2\pi, \quad q < p, \tag{36}$$

where $p, q \in \mathbb{N}$ are coprime numbers. We denote

$$G_p = \left\{ \frac{2\pi k}{p}, \; k = 1, 2, \ldots, p \right\}, \tag{37}$$

$$\Delta = \left(\frac{9\pi}{20}, \frac{11\pi}{20} \right) \cup \left(\frac{29\pi}{20}, \frac{31\pi}{20} \right). \tag{38}$$

It is well known that

$$\{(kx)/2\pi : \; k = n+1, n+2, \ldots, n+p\} = G_p, \tag{39}$$

for any $n \in \mathbb{N}$. We suppose $\nu(x) = p$. To prove (35) for x defined in (36), it is enough to show that

$$\#(G_p \cap \Delta) \le \frac{2p}{5}, \quad p \ne 4. \tag{40}$$

The case $p = 4$ is excluded because of (34). Since $G_p \cap \Delta = \varnothing$ if $p = 2, 3, 5, 6$, then (40) holds for such p's. If $p \ge 7$, then rough estimation gives

$$\# \left(G_p \cap \left(\frac{9\pi}{20}, \frac{11\pi}{20} \right) \right) < \frac{p}{20} + 1,$$

$$\# \left(G_p \cap \left(\frac{29\pi}{20}, \frac{31\pi}{20} \right) \right) < \frac{p}{20} + 1,$$

which implies

$$\#(G_p \cap \Delta) < 2\left(\frac{p}{20} + 1 \right) < \frac{2p}{5}, \quad p \ge 7. \tag{41}$$

In the case of irrational $x/2\pi$ we shall use an approximation of $x/2\pi$ by rational numbers. It is known that there exist coprime numbers l and ν, with $\nu > 40$, such that

$$\left| x - \frac{2\pi l}{\nu} \right| < \frac{2\pi}{\nu^2}. \tag{42}$$

We consider the set

$$\Delta_0 = \left(\frac{8\pi}{20}, \frac{12\pi}{20} \right) \cup \left(\frac{28\pi}{20}, \frac{32\pi}{20} \right) \supset \Delta.$$

Similarly as in (41), we get

$$\#((a + G_\nu) \cap \Delta_0) < 2\left(\frac{\nu}{10} + 1 \right) < \frac{2\nu}{5}, \quad \nu > 40, \tag{43}$$

for arbitrary $a \in \mathbb{R}$, where

$$a + G_\nu = \left\{ \left(a + \frac{2\pi lk}{\nu} \right) /2\pi,\ k = 1, 2, \ldots, \nu \right\}.$$

From (42) it follows that

$$\left| kx - \frac{2\pi lk}{\nu} \right| < \frac{2\pi k}{\nu^2} \le \frac{2\pi}{\nu} < \frac{\pi}{20}, \quad k = 1, 2, \ldots, \nu,$$

and therefore we conclude, that the condition

$$(a + kx)/2\pi \in \Delta, \quad k = 1, 2, \ldots, \nu, ,$$

implies

$$\left(a + \frac{2\pi lk}{\nu} \right) /2\pi \in \Delta_0.$$

Taking $a = nx$, thus we obtain

$$\begin{aligned} \#\{k \in \mathbb{N} : n < k \le n + \nu, \quad (kx)/2\pi \in \Delta\} \\ &= \#\{k \in \mathbb{N} :\ 1 \le k \le \nu, \quad (a + kx)/2\pi \in \Delta\} \\ &= \#\{k \in \mathbb{N} :\ 1 \le k \le \nu, \quad \left(a + \frac{2\pi lk}{\nu} \right) /2\pi \in \Delta_0\} \\ &= \#((a + G_\nu) \cap \Delta_0) < \frac{2p}{5} \end{aligned} \tag{44}$$

and the lemma is proved. □

Lemma 5. *If a sequence x_n, $n = 1, 2, \ldots$, satisfies the conditions (5), then for any sequence of coefficients a_n, with (6), the series*

$$\sum_{n=1}^{\infty} a_n \cos(x_n) \tag{45}$$

is divergent.

Proof. Applying Lemma 3, we may find sequences of numbers $p_n, q_n \in \mathbb{N}$, $q_n > p_n > n$, $n = 1, 2, \ldots$, such that

$$0 < (x_j)/2\pi < \frac{\pi}{50}, \quad p_n \le j \le q_n, \tag{46}$$

$$\sum_{j=p_n}^{q_n} \sqrt{\Delta^2 x_j} > 10^{-3}. \tag{47}$$

From (6) and (47) we obtain

$$\sum_{j=p_n}^{q_n} a_j \ge c \sum_{j=p_n}^{q_n} \sqrt{\Delta^2 x_j} > c \cdot 10^{-3},$$

then, combining this with (46), we get

$$\sum_{j=p_n}^{q_n} a_j \cos(x_j) > \cos(\pi/50) \sum_{j=p_n}^{q_n} a_j > c \cdot \cos(\pi/50) \cdot 10^{-3}.$$

This implies the divergence of series (45). □

3. Proof of theorems

Proof of Theorem 1. The real part of series (7) is

$$\sum_{n=1}^{\infty} a_n \cos(nx + \rho_n). \tag{48}$$

Defining

$$x_n = nx + \rho_n,$$

we have $\Delta^2 x_n = \Delta^2 \rho_n$. So x_n satisfies the same conditions (5) and (6) as ρ_n. Therefore the divergence of series (48) immediately follows from Lemma 5. The imaginary part of (7) is the series

$$\sum_{n=1}^{\infty} a_n \sin(nx + \rho_n) = \sum_{n=1}^{\infty} a_n \cos(nx + \rho_n - \pi)$$

and its divergence can be obtained similarly, discussing the sequence $x_n = nx + \rho_n - \pi$. □

Proof of Theorem 2. We fix an arbitrary number

$$x \neq \frac{\pi}{2} \mod 2\pi \tag{49}$$

and denote

$$x_n = \pi\rho_n - nx - \frac{\pi}{2}. \tag{50}$$

Applying Lemma 3 we may define sequences $q_n > p_n > n$ with conditions (46) and (47). Observe, that if

$$p_n \le k \le q_n \text{ and } |\cos kx| \ge \sin(\pi/20), \tag{51}$$

then

$$(-1)^{[\rho_k]} \cos kx = |\cos kx|. \tag{52}$$

Indeed, from (46) and (51) we have

$$(x_k)/2\pi \in (0, \pi/50),$$
$$(kx)/2\pi \notin \Delta = \left(\frac{9\pi}{20}, \frac{11\pi}{20}\right) \cup \left(\frac{29\pi}{20}, \frac{31\pi}{20}\right),$$

which together with (50) implies

$$\begin{aligned}\operatorname{sign}(\cos kx) &= \operatorname{sign}(\cos(kx+x_k))\\ &= \operatorname{sign}(\cos(\pi\rho_k-\pi/2)) = \operatorname{sign}(\sin\pi\rho_k) = (-1)^{[\rho_k]},\end{aligned}$$

and therefore we get (52). According to Lemma 4 there exists a number $\nu=\nu(x)$, satisfying (35). Hence, from (52) we obtain

$$\begin{aligned}&\#\{k\in\mathbb{N}:\ m<k\le m+\nu,\ (-1)^{[\rho_k]}\cos kx\ge\sin(\pi/20)\}\\ &\quad=\#\{k\in\mathbb{N}:\ m<k\le m+\nu,\ |\cos kx|\ge\sin(\pi/20)\}\ge\frac{3\nu}{5}\end{aligned}\tag{53}$$

provided we have $p_n\le m$, $m+\nu\le q_n$. We let

$$s_n=\left[\frac{p_n}{\nu}\right]+1,\quad t_n=\left[\frac{q_n}{\nu}\right].$$

We have

$$(s_n-1)\nu\le p_n<s_n\nu,\quad t_n\nu\le q_n<(t_n+1)\nu.\tag{54}$$

Then, denoting

$$G_k=\{j\in\mathbb{N}:\ k\nu<j\le(k+1)\nu,\ (-1)^{\left[\frac{\rho_j}{\pi}\right]}\cos jx\ge\sin(\pi/20)\}$$

from (53) we obtain

$$\#G_k\ge\frac{3\nu}{5},\quad s_n\le k<t_n.$$

Hence we conclude, if

$$\begin{aligned}&\sum_{j=\nu k+1}^{\nu(k+1)}(-1)^{[\rho_j]}a_j\cos jx\\ &\qquad\ge\sum_{j\in G_k}(-1)^{[\rho_j]}a_j\cos jx-\frac{2\nu}{5}\sin(\pi/20)a_{\nu k}\\ &\qquad\ge\sin(\pi/20)\left(\frac{3\nu}{5}a_{\nu(k+1)}-\frac{2\nu}{5}a_{\nu k}\right)\\ &\qquad=\frac{\nu\sin(\pi/20)}{5}\left(3a_{\nu(k+1)}-2a_{\nu k}\right),\end{aligned}\tag{55}$$

provided $s_n\le k<t_n$. From (47) and the relation $\Delta^2x_n=\Delta^2\rho_n\to 0$ it follows that the inequality

$$\sum_{j=\nu s_n+1}^{\nu t_n}a_j\ge c\sum_{j=\nu s_n+1}^{\nu t_n}\sqrt{\Delta^2x_j}\ge c\cdot 10^{-4}\tag{56}$$

holds for sufficiently large $n > n_0$. From (55) and (56) we obtain

$$\begin{aligned}
\Big|\sum_{j=\nu s_n+1}^{\nu t_n}(-1)^{[\rho_j]}a_j\cos jx\Big| \\
&= \left|\sum_{k=s_n}^{t_n-1}\sum_{j=\nu k+1}^{\nu(k+1)}(-1)^{[\rho_j]}a_j\cos jx\right| \\
&> \frac{\nu\cdot\sin(\pi/20)}{5}\left(3\sum_{k=s_n}^{t_n-1}a_{\nu(k+1)}-2\sum_{k=s_n}^{t_n-1}a_{\nu k}\right) \\
&= \frac{\nu\cdot\sin(\pi/20)}{5}\left(\sum_{k=s_n+1}^{t_n-1}a_{\nu k}+3a_{\nu t_n}-2a_{\nu s_n}\right) \\
&\geq \frac{\sin(\pi/20)}{5}\left(\sum_{j=\nu(s_n+1)+1}^{\nu(t_n-1)}a_j+\nu(3a_{\nu t_n}-2a_{\nu s_n})\right)
\end{aligned}$$

Using (56) and (54), we obtain

$$\limsup_{n\to\infty}\left|\sum_{j=\nu s_n+1}^{\nu t_n}(-1)^{[\rho_j]}a_j\cos jx\right| > 0.$$

So the series (8) is divergent if (49) holds. If $x = \pi/2$, then we have

$$\sum_{n=0}^{\infty}(-1)^{[\rho_n]}a_n\cos nx = \sum_{k=0}^{\infty}(-1)^{[\rho_{2k}]}a_{2k}\cos k\pi$$

The last series is divergent because ρ_{2k} again satisfies the same conditions (5) and the divergent of such series at $x = \pi$ has been already proved. This completes the proof of theorem. □

References

1. N. Bary, A treatise on trigonometric series, Pergamon Press, (1964).
2. L. Carleson, On the convergence and growth of partial sums of Fourier series, Acta Math., 116(1966), 135-167.
3. A. Dvoretzky, P. Erdos, On power series which diverges everywhere on the unit circle of convergence, Michigan Math. Journal, 3(1955-1956), No 1, 31-35.
4. P. Fatou, Series trigonometriques et series de Taylor, Acta Math., 30(1906), 335-400.
5. S. Sh. Galstyan, Everywhere divergent trigonometric series, Math. Notes, 37(1985), No 2, 105-108.
6. G. H. Hardy, J. E. Littlewood, Some problems of diophantine approximation: A remarkable trigonometrical series, Nat. Acad. Sci. U.S.A., 2(1916), 583-586.

7. F. Herzog, A note on power series which diverge everywhere on the unit circle of convergence, Michigan Math. Journal, 3(1955-1956), 3, No 1, 31-35.
8. N. N. Luzin, On a case of Taylor series, Mat. Sbornik, 23(1912), 295-302.
9. L. Neder, ZüTheorie der trigonometrischen. Reihen-Math. Ann., 84(1921), 117-136.
10. S. B. Stechkin, On convergence and divergence of trigonometric series, Math. Survey, 6(1951), No 2, 148-149.
11. H. Steinhaus, Une trigonometrique partout divergente, C.R. de la Soc. Sci. de Varsovie, (1912).
12. P. L. Ul'janov, Solved and unsolved problems in the theory of trigonometric and orthogonal series, Math. Survey, 119(1964), No 1(115), 369.

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