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Degenerate Third Order Differential-operator Equations

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Abstract
We consider the boundary value problem (BVP) for degenerate third order differential-operator equation. The study of linear differential equations, coefficients of which are, in general, unbounded operators in the Hilbert or Banach spaces, is expedient not only because they contain many differential equations with partial derivatives, but also we get the opportunity to look from a single point of view at ordinary differential operators, as well as at the operators with partial derivatives. First we consider one-dimensional case of the BVP. Then it is proved that under certain conditions on the corresponding operators and on the degeneration index, a generalized solution of the corresponding operator-differential equation exists and is unique.

Keywords Degenerate equation · Hilbert space · Linear operator · Complete system · Generalized solution

Introduction

In the presented paper we consider the following boundary value problem (BVP) for degenerate differential-operator equation of third order

$$Su \equiv -(t^{\alpha}u'')' - Au' + Pu = f, \tag{1.1}$$

with boundary conditions $u(0) = u'(0) = u'(b) = 0$, where $t \in (0, b)$, $\alpha \geq 0$, $\alpha \neq 1$, $\alpha < \frac{5}{2}$.
Linear operators $A, P : H \to H$ are in general unbounded operators in some *separable* Hilbert space H and commute with $D_t \equiv \frac{d}{dt}$, $f \in L_2((0, b), H)$, i.e.,

$$\int_0^b \|f\|^2_{L_2((0,b),H)}\, dt < \infty.$$

Let us give a short historical overview. First note the important article and the book by A.A. Dezin (see [2, 3]), where the second order degenerated differential-operator equations are considered. Then author published an article for the degenerate differential-operator equation of fourth order (see [11]). It is worth to note also the articles of the author (see [12, 13]) and the article of B.-W. Schulze with L. Tepoyan ([9]). Note also the book by J. Weidmann (see [15]) and the articles of N. Yataev ([16–18]).
We prove that under some conditions on the operators A, P and α the BVP (1.1) has unique generalized solution $u \in L_2((0, b), H)$ for arbitrary $f \in L_2((0, b), H)$.

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We suppose that the linear operators $A, P : H \to H$ have *common complete system* of eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$, which form Riesz basis in H, i.e., $A\varphi_k = a_k\varphi_k$, $P\varphi_k = p_k\varphi_k$, $k \in \mathbb{N}$ and for every $x \in H$ we have the following representation

$$x = \sum_{1}^{\infty} x_k\varphi_k.$$

Moreover, there are some positive *constants* $c_1 > 0, c_2 > 0$ such that for any $x \in H$ we have

$$c_1 \sum_{k=1}^{\infty} |x_k|^2 \le \|x\|_H^2 \le c_2 \sum_{k=1}^{\infty} |x_k|^2. \tag{1.2}$$

First we investigate *one-dimensional* case of the operator equation (1.1), when $Au = au$, $Pu = pu$, $a, p \in \mathbb{C}$. Then we consider general operator equation and prove the existence and uniqueness of the generalized solution.
Note also that for even order degenerated differential equations we usually define generalized solutions in weighted Sobolev spaces, which is very useful. For differential equations of odd order there is no such possibility and therefore we have to find some way to prove the existence and uniqueness of the generalized solution.

One-dimensional case

In this section we consider *one-dimensional case* of BVP (1.1), i.e. when $Su = au$

$$Su \equiv -(t^{\alpha}u'')' - au' + pu = f, \quad u(0) = u'(0) = u'(b) = 0, \tag{2.1}$$

where $f \in L_2(0, b)$.
Our goal is to find such values of α, a and p, for which BVP (2.1) has unique generalized solution for any $f \in L_2(0, b)$ as well to describe domain of definition for operator S.
We will first give description of the *domain of definition* for differential expression

$$S_1u \equiv -(t^{\alpha}u'')' - au' = f, \tag{2.2}$$

since the term pu evidently belongs to $D(S) = D(S_1)$. Denote $u' = v$. Then we obtain new BVP, namely

$$-(t^{\alpha}v')' - av = f, \quad v(0) = v(b) = 0. \tag{2.3}$$

BVP (2.3) were considered in the article of A.A. Dezin (see [2]) forthe degenerate second order ordinary differential equation. It was proved that for $a \le 0$ the BVP (2.3) is uniquely solvable for every $f \in L_2(0, b)$.
Let us first consider the case $a = 0$. Then we obtain

$$u'(t) = v(t) = -\int_0^t \tau^{-\alpha}\Big(\int_0^{\tau} f(\eta)d\eta\Big)\, d\tau + c_1t^{1-\alpha} + c_2.$$

Now, using the inequality of Cauchy we obtain the following inequality for some constant $c > 0$

$$|u(t| \le ct^{\frac{5}{2}-\alpha}, \tag{2.4}$$

where the constant c depends from the function f. In the same way we can estimate $u(t)$ in the case $a \neq 0$ and again we obtain the same inequality (2.4). Therefore, we will suppose that $\frac{5}{2} - \alpha > 0$, i.e. $0 \le \alpha < \frac{5}{2}, \alpha \neq 1$.
Let us consider as an example the following boundary value problem: *a self-adjoint equation of the second type of the fourth order*.
Here we consider special case of the fourth order one-dimensional equation

$$Su \equiv D_t^2(t^{\alpha}D_t^2)u - qD_t^2u = f, 0 \le \alpha \le 4, q = \mathtt{const}, \alpha \neq 1, \alpha \neq 3. \tag{2.5}$$

Definition 3. We say that $u \in \dot{W}_2^\alpha(0, b)$ is a generalized solution of the equation (2.5), when

$$\{u, v_h\}_\alpha + q(D_t u, D_t v_h) = (f, v_h), \tag{2.6}$$

for every $v \in W$ and $h > 0$, were $v_h = v\psi_h$.
Theorem 4. If $f \in L_2(0, b)$, then the equation (2.5) has a unique generalized solution for $q > 0$ and $0 \le \alpha < 3$.
Proof. Existence. Let $0 \le \alpha < 3$. Solving the homogeneous equation and applying the method of variation of constants, we obtain the following explicit representation of the solution for the equation (2.6)

$$u(t) = k_1^{-1}\left\{ \int_0^t \left[w_2(\tau) \int_0^\tau w_1(\eta)F(\eta)\, d\eta + w_1(\tau) \int_\tau^b w_2(\eta)F(\eta)\, d\eta \right] d\tau \right\}, \tag{2.7}$$

where

$$F(t) = \int_0^t f(\tau)\, d\tau + k_2, \quad w_1(t) = t^{(1-\alpha)/2} K_\nu(x),$$

$$w_2(t) = t^{(1-\alpha)/2}(I_\nu(x) + \beta K_\nu(x)), \quad x = 2(\alpha-2)^{-1} q^{1/2} t^{(2-\alpha)/2},$$

and $I_\nu(x)$ and $K_\nu(x)$ are Bessel functions of purely imaginary argument of order $\nu = (\alpha-1)(\alpha-2)^{-1}$, β is determined by the condition

$$I_\nu|_{t=b} + \beta K_\nu|_{t=b} = 0,\ k_1 = b^\alpha w(b).$$

Here $w(t)$ is the Wronskian of the functions w_1 and w_2, k_2 is a constant determined by the condition $u|_{t=b} = 0$. It follows from the asymptotic expansions of $I_\nu(x)$ and $K_\nu(x)$, that $u(t)$, given by (2.7), is in $W_\alpha^2(0, b)$. Thus we have the equality

$$\{u, v_h\}_\alpha = (D_t^2 t^\alpha D_t^2 u, v_h),$$

obtained by integration by parts. For the details see [11].
Let us now prove the uniqueness of the generalized solution. We have to prove that the relation

$$\{u, v_h\}_\alpha + q(D_t u, D_t v_h) = 0,$$

which holds for some $u \in \dot{W}_\alpha^2$ and all $v \in \dot{W}_\alpha^2$, implies that $u = 0$.
It follows from (11) that $u(t)$ is a weak solution of the homogeneous equation corresponding to (8), and is therefore a classical solution on each subinterval of V_t [9]. First, let l_1 and d_1 be sufficiently close to 0 and b, and vary l and d so that $0 < l \le l_1 < d_1 \le d < b$. Let $v_1(t) = t^{(1-\alpha)/2} I_\nu(x)$, and let $v_2(t) = t^{(1-\alpha)/2} K_\nu(x)$. On $[l_1, d_1]$ we have

$$u(t) = c_1 \int_{l_1}^t v_1(\tau)\, d\tau + c_2 \int_{l_1}^t v_2(\tau)\, d\tau + c_3 \int_{l_1}^t v_1(\tau) \int_{l_1}^t v_2(\eta)\, d\eta\, d\tau +$$
$$+ c_3 \int_{l_1}^t v_2(\tau) \int_\tau^{d_1} v_1(\eta)\, d\eta\, d\tau + c_4,$$

where c_i, $i = 1, 2, 3, 4$ are constants.
On $[l, d]$ the same function has a similar representation with coefficients $c_i^{l,d}$, $i = 1, 2, 3, 4$. Replacing l_1 and d_1 by l and d, and using the coincidence of the two representations for $t \in [l_1, d_1]$, we obtain

$$c_3^{l,d} = c_3, \quad c_1^{l,d} = c_1 - c_3 \int_l^{l_1} v_2(\eta)\, d\eta, \quad c_2^{l,d} = c_2 - c_3 \int_{d_1}^d v_1(\eta)\, d\eta,$$

$$c_4^{l,d} = c_4 - c_3 \int_l^{l_1} v_1(\tau) \int_l^\tau v_2(\eta)\, d\eta\, d\tau - c_3 \int_l^{l_1} v_2(\tau) \int_\tau^d v_1(\eta)\, d\eta\, d\tau -$$

$$-\left[c_1 - c_3 \int_l^{l_1} v_2(\eta)\,d\eta\right] \int_l^{l_1} v_1(\tau)\,d\tau - \left[c_2 - c_3 \int_{d_1}^{d} v_1(\eta)\,d\eta\right] \int_l^{l_1} v_2(\tau)\,d\tau.$$

Letting l and d tend to 0 and b, and using the fact that $u \in \dot{W}_\alpha^2$ and $0 \le \alpha < 3$, we find that

$$\int_l^d t^\alpha |D_t^2 u|^2\,dt \le C$$

uniformly with respect to l and d, and $u|_{t=0} = u|_{t=b} = D_t u|_{t=b} = 0$. These conditions can be satisfied only if the following relations hold:

$$c_1 - c_3 \int_0^{l_1} v_2(\tau)\,d\tau = 0,$$

$$c_2 v_2(b) + c_3 v_1(b) \int_0^b v_2(\tau)\,d\tau - c_3 v_2(b) \int_{d_1}^b v_1(\tau)\,d\tau = 0,$$

$$c_4 - c_3 \int_0^{l_1} v_1(\tau) \int_0^\eta v_2(\eta)\,d\eta\,d\tau - c_3 \int_0^{l_1} v_2(\tau) \int_\tau^{d_1} v_1(\eta)\,d\eta\,d\tau - c_2 \int_0^{l_1} v_2(\tau)\,d\tau = 0, \quad (12)$$

$$c_4 + c_3 \int_0^{l_1} v_2(\eta)\,d\eta \int_{l_1}^b v_1(\tau)\,d\tau + \left[c_2 - c_3 \int_{d_1}^b v_1(\eta)\,d\eta\right] \int_{l_1}^b v_2(\tau)\,d\tau +$$

$$+c_3 \int_{l_1}^b v_1(\tau) \int_{l_1}^\tau v_2(\eta)\,d\eta\,d\tau + c_3 \int_{l_1}^b v_2(\tau) \int_\tau^b v_1(\eta)\,d\eta\,d\tau = 0.$$

Hence $u(t) = 0$ on each interval $[l, d]$, since the determinant of the system (12) does not vanish, and so $c_i = 0,\ i = 1, 2, 3, 4.$

Differential-operator equations

In this section we consider the boundary value problem for differential-operator equation

$$Su \equiv -(t^\alpha u''(t))' - Au' + Pu = f(t), \tag{3.1}$$

with boundary conditions $u(0) = u'(0) = u(b) = 0$, where $t \in (0, b), \alpha \ge 0,\ A, P : H \to H$ are linear operators in some separable Hilbert space H and $f \in L_2((0, b), H)$. We suppose that the operators A and P have common complete system of eigenfunctions $\{\varphi_k\}_{k=1}^\infty$, which form Riesz basis in H (see [11, 12]). Therefore, we can write

$$u(t) = \sum_{k=1}^\infty u_k(t)\varphi_k, \tag{3.2}$$

$$f(t) = \sum_{k=1}^\infty f_k(t)\varphi_k. \tag{3.3}$$

Then instead of operator equation (3.1) with boundary conditions

$$u(0) = u'(0) = u'(b) = 0$$

we obtain infinite chain of ordinary differential equations

$$S_k u_k \equiv -(t^\alpha u_k''(t))' - a_k u_k' + p_k u_k = f_k(t), k \in \mathbb{N} \tag{3.4}$$

with boundary conditions

$$u_k(0) = u_k'(0) = u_k'(b) = 0, k \in \mathbb{N}.$$

It follows from the considerations above that for the generalized solution $u \in L_2((0,b), \mathcal{H})$ we obtain the following inequality

$$\|u\|_{L_2((0,b),H)} \leq c\|f\|_{L_2((0,b),H)}.$$

Let us give an interesting example (see [2]).
Example 1. Let $V_x \equiv [0, 2\pi]^3 \subset \mathbb{R}^3$ and the operator A be the closure in the space $L_2(V_x)$ of the differential operation

$$A(-iD) \equiv -iD_1 - i\beta D_2 + D_3 - i\gamma D_2^2,$$

originally defined on the smooth functions with the periodicity conditions with respect to each variable x_k with the period 2π, β and γ be some irrational numbers. Then the point spectrum of the operator $A : L_2(V_x) \to L_2(V_x)$ will consist of the points in the complex plane $\mathbb{C}$ having the coordinates

$$A(k) = k_1 + \beta k_2 + i(k_3 + \gamma k_2^2), \quad k = (k_1, k_2, k_3).$$

The set of values of the polynomial $A(k), k \in \mathbb{Z}$ is dense in the complex plane $\mathbb{C}$. This follows from the uniform distribution in the unit square of the fractional parts of the pair $(\beta k_2, \gamma k_2^2), k_2 \in \mathbb{Z}$. Hence we get that the spectrum of the operator A (closure of the set $A(k)$, $k \in \mathbb{Z}^3$) coincides with the whole complex plane. Note that in this example the role of the functions $\{\varphi_k\}_{k=1}^\infty$ played the functions $e^{-i(k_1x_1+k_2x_2+k_3x_3} = e^{-ik\cdot x}, k \in \mathbb{Z}^3$, which form an orthogonal basis in $L_2(V_x)$. We prove that the spectrum $\sigma(P)$ of the operator P in the case of the operator $A : L_2(V_x) \to L_2(V_x)$ is the whole complex plane $\mathbb{C}$, i.e. $\rho(P) = \emptyset$. In fact, let $\lambda_m, m \in \mathbb{N}$ be the eigenvalues of the operator P. Then the numbers $\lambda_m + A(k), m \in \mathbb{N}, k \in \mathbb{Z}^3$ are the eigenvalues for the operator P. Obviously, this set is dense in $\mathbb{C}$ because of the density of $A(k), k \in \mathbb{Z}^3$ in $\mathbb{C}$, i.e. $\sigma(P) = \mathbb{C}$, since the spectrum is closed.
Now, we define the so-called $\prod$-operators, which are very useful.
Let V be cube $(0, 2\pi)^n$ in R^n. To each polynomial A with constant complex coefficients we associate differential operation $A(-iD)$ such that (here $|\alpha| = \alpha_1 + ... + \alpha_n$)

$$s^\alpha = s_1^{\alpha_1} \cdots s_n^{\alpha_n},$$

$$A(-iD)e^{is\cdot x} = A(s)e^{is\cdot x}.$$

We call the corresponding operator $A : H \to H$ $\prod$-operator.
Let us formulate some properties of $\prod$-operators without proofs (the proofs can be found, for example, from the monograph of A.A. Dezin "General Questions of the Theory of Boundary Value Problems").
Proposition 1. Each two $\prod$-operators A_1 and A_2 commute.
Proposition 2. Each $\prod$-operator A is normal.
Let us formulate other propositions concerning $\prod$-operators.
Proposition 3. For $n \leq 2$ the resolvent set $\rho(A)$ for $\prod$-operator A is nonempty.
For the case $n > 2$ there exist operators A, for which the spectrum coincides with $\mathbb{C}$.
Proposition 4. Operator $A^{-1} : H \to H$, inverse for some $\prod$-operator A, is compact only then, when

$$|A(s)| \to \infty$$

Example 2. (see A.A. Dezin, [2])
Let V be cube $(0, 2\pi)^3$ in $\mathbb{R}^3$,

$$A_k(D) \equiv (-1)^k(D_1^2 + D_2^2 - 1), k = 1, 2,$$

$$P(D) = D_1^2 - D_3^4, D_l \equiv \frac{\partial}{\partial x_l},$$

under conditions of periodicity with respect to all x_i with period 2π. Then the spectrum of the operators consists from the points $(-1)^{k+1}(s_1^2 + s_2^2 + 1)$ and $-s_1^2 - s_3^4$, where $s_l \in \mathbb{Z}$.
Thus we immediately obtain the following result.
Proposition 5. Generalized solution of the differential-operator equation

$$\big(- D_t^\alpha D_t + (-1)^k(D_1^2 + D_2^2 - 1)D_t - D_1^2 + D_3^4)u = f$$

in domain $[0, b] \times V$ under conditions of the periodicity with respect x and conditions $u|_{t=0} = 0, u|_{t=b} = 0$ with respect t exists and is unique.
Now, we define the so-called model operators (see [4]).
Definition 1. The operator $A : \mathbb{H} \to \mathbb{H}$ is called M operator, when $\varphi_k, k \in \mathbb{N}$ is the system of eigenfunctions of the operator A, which form Riesz basis in $\mathbb{H}$.
Thus we can write

$$u = \sum_{k=1}^{\infty} u_k \varphi_k,$$

$$F(A)u = \sum_{s=1}^{\infty} F(A_s)u_s e^{is\cdot x}.$$

Proposition 6. The spectrum $\sigma(F)$ of the operator F consists from the closure in the complex plane $F(S)$, which forms point spectrum $P\sigma(F)$ of operator P. The set

$$C\sigma(F) = \sigma(F) \setminus P\sigma(F)$$

forms continuous spectrum of the operator F.
Now, we define tensor product of separable Hilbert spaces $\mathbb{H}'$ and $\mathbb{H}''$, where $\{\phi_k\}_{k=1}^\infty$ and $\{\psi_k\}_{k=1}^\infty$ are given orthogonal bases. Let us form Hilbert space $\mathbb{H}$ in the following way. We take the pairs $\phi_k \otimes \psi_j, k, j \in \mathbb{N}$ as $\mathbb{H}$. Let us define scalar product in the following way

$$(\phi_k \otimes \psi_j, \phi_l \otimes \psi_i) = (\phi_k, \phi_l)(\psi_j, \psi_i).$$

Thus we form new Hilbert space, which is called tensor product of Hilbert spaces $\mathbb{H}'$ and $\mathbb{H}''$, which we will denote by

$$\mathbb{H} = \mathbb{H}' \bigotimes \mathbb{H}''.$$

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Declarations

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