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published in *Proceedings of Artsakh State University*, (2013), No. 1(27), pp. 12–20.

ON THE "MOVEMENT" OF THE ZEROS OF EIGENFUNCTIONS OF THE STURM-LIOUVILLE PROBLEM

TIGRAN HARUTYUNYAN, AVETIK PAHLEVANYAN, YURI ASHRAFYAN

Abstract. We study the dependence of the zeros of eigenfunctions of Sturm-Liouville problem on the parameters that define the boundary conditions. As a corollary, we obtain Sturm oscillation theorem, which states that the n-th eigenfunction has n zeros.

Let us consider Sturm-Liouville boundary value problem $L(q,\alpha,\beta)$:

$$ly \equiv -y'' + q(x)\, y = \mu y,\ 0 < x < \pi,\ \mu \in \mathbb{C}, \tag{1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0,\ \alpha \in (0,\pi], \tag{2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0,\ \beta \in [0,\pi), \tag{3}$$

where $q \in L^1_{\mathbb{R}}[0,\pi]$, i.e. q is a real-valued, summable function on $[0,\pi]$.

By $L(q,\alpha,\beta)$ we also denote the self-adjoint operator corresponding to the problem (1)–(3).

It is well-known, that the problem $L(q,\alpha,\beta)$ has a countable set of simple, real eigenvalues (see, e.g. [1, 2, 3, 4]), which we denote by $\mu_n(q,\alpha,\beta)$, $n = 0,1,\ldots,$ (emphasizing the dependence on q, α and β) and enumerate in increasing order:

$$\mu_0(q,\alpha,\beta) < \mu_1(q,\alpha,\beta) < \cdots < \mu_n(q,\alpha,\beta) < \ldots . \tag{4}$$

In the papers [4, 7] it was introduced the concept of the eigenvalues function (EVF) of a family of Sturm-Liouville operators $\{L(q,\alpha,\beta);\alpha \in (0,\pi], \beta\,[0,\pi)\}$. For fixed q it is a function of two variables γ and δ $(\gamma = \alpha + \pi n \in (0,\infty), \delta = \beta - \pi m \in (-\infty,\pi))$ determined through the eigenvalues $\mu_n(q,\alpha,\beta)$, $n = 0,1,2,\ldots,$ by the following formula:

$$\mu(\gamma,\delta) = \mu(\alpha + \pi n, \beta - \pi m) := \mu_{n+m}(q,\alpha,\beta),\ n,m = 0,1,2,\ldots. \tag{5}$$

It is proved that this function is analytic with respect to γ and δ. It is strictly increasing by γ and strictly decreasing by δ.

It is also known (see, e.g. [1]) that every nontrivial solution $y(x,\mu)$ of the equation (1) may have only simple zeros (if $y(x_0,\mu) = 0$, then $y'(x_0,\mu) \neq 0$), and (see, e.g. [5]) every solution $y(x,\mu)$ is a continuously differentiable function with respect to the totality of variables x and μ. Therefore, by applying the implicit function theorem (see, e.g. [6, p. 452]), we get that the zeros of the solution $y(x,\mu)$ are the continuously differentiable functions with respect to μ. Since the function $x = x(\mu)$, such that the identity $y(x(\mu),\mu) \equiv 0$ is true for all μ from some interval

2010 *Mathematics Subject Classification.* 34B24, 34C10.
Key words and phrases. Sturm-Liouville problem, zeros of eigenfunctions.

(a,b), is called the solution of the equation $y(x,\mu)=0$, then differentiating the last identity with respect to μ we obtain:

$$\frac{dy(x(\mu),\mu)}{d\mu}=\frac{\partial y(x(\mu),\mu)}{\partial x}\frac{dx(\mu)}{d\mu}+\frac{\partial y(x(\mu),\mu)}{\partial \mu}\equiv 0,\ \mu\in(a,b). \tag{6}$$

Let us denote $\dfrac{\partial y(x,\mu)}{\partial\mu}:=\dot{y}(x,\mu)$ and write the identity (6) in the following form:

$$\frac{dx(\mu)}{d\mu}=\dot{x}(\mu)=-\frac{\dot{y}(x(\mu),\mu)}{y'(x(\mu),\mu)},\ \mu\in(a,b). \tag{7}$$

On the other hand, let us write down the fact that $y(x,\mu)$ is the solution of the equation (1), i.e.

$$-y''(x,\mu)+q(x)\,y(x,\mu)\equiv\mu y(x,\mu),\ 0<x<\pi,\ \mu\in\mathbb{C}, \tag{8}$$

and differentiating this identity with respect to μ we receive:

$$-\dot{y}''(x,\mu)+q(x)\,\dot{y}(x,\mu)\equiv y(x,\mu)+\mu\dot{y}(x,\mu). \tag{9}$$

Multiplying (8) by $\dot{y}$, (9) by y and subtracting from the second obtained identity the first one, we get

$$y''(x,\mu)\,\dot{y}(x,\mu)-\dot{y}''(x,\mu)\,y(x,\mu)\equiv y^2(x,\mu),\ 0<x<\pi,\ \mu\in\mathbb{C},$$

i.e.

$$\frac{d}{dx}\left[y'(x,\mu)\,\dot{y}(x,\mu)-\dot{y}'(x,\mu)\,y(x,\mu)\right]\equiv y^2(x,\mu). \tag{10}$$

If we integrate this identity with respect to x from 0 to a $(0\le a\le\pi)$, then

$$\begin{aligned}y'(a,\mu)\,\dot{y}(a,\mu)-\dot{y}'(a,\mu)\,y(a,\mu)-y'(0,\mu)\,\dot{y}(0,\mu)+&\\ +\dot{y}'(0,\mu)\,y(0,\mu)=\int_0^a y^2(x,\mu)\,dx,&\end{aligned} \tag{11}$$

and if we integrate with respect to x from a to π, then we get

$$\begin{aligned}y'(\pi,\mu)\,\dot{y}(\pi,\mu)-\dot{y}'(\pi,\mu)\,y(\pi,\mu)-y'(a,\mu)\,\dot{y}(a,\mu)+&\\ +\dot{y}'(a,\mu)\,y(a,\mu)=\int_a^\pi y^2(x,\mu)\,dx.&\end{aligned} \tag{12}$$

Now, as $y(x,\mu)$ let us take $y=\varphi(x,\mu,\alpha,q)-$ the solution of the equation (1), satisfying the following initial conditions:

$$\varphi(0,\mu,\alpha,q)=\sin\alpha,\quad \varphi'(0,\mu,\alpha,q)=-\cos\alpha. \tag{13}$$

It is easy to see that eigenfunctions of the problem $L(q,\alpha,\beta)$ are obtained from the solution $\varphi(x,\mu,\alpha,q)$ at $\mu=\mu_n(q,\alpha,\beta)$ (here we use (5)), i.e.

$$\begin{aligned}\varphi_n(x,q,\alpha,\beta)&:=\varphi_n(x)=\varphi(x,\mu_n(q,\alpha,\beta),\alpha,q)=\\ &=\varphi(x,\mu(\alpha+\pi n,\beta),\alpha,q)=\varphi(x,\mu(\alpha,\beta-\pi n),\alpha,q)=\\ &=\varphi(x,\mu(\alpha,\delta),\alpha,q)|_{\delta=\beta-\pi n}=\varphi(x,\mu(\alpha,\delta))|_{\delta=\beta-\pi n}=\\ &=\varphi(x,\mu(\alpha,\beta-\pi n)).\end{aligned} \tag{14}$$

Let $0\le x_n^0<x_n^1<\dots<x_n^m\le\pi$ be the zeros of the eigenfunction $\varphi_n(x,q,\alpha,\beta)=\varphi(x,\mu(\alpha,\beta-\pi n))$, i.e. $\varphi_n\left(x_n^k,q,\alpha,\beta\right)=\varphi\left(x_n^k,\mu(\alpha,\beta-\pi n)\right)=0$, $k=0,1,\dots,m$.

Let q,α,n be fixed. We consider the following questions:

a) How are the zeros $x_n^k = x_n^k(\beta)$, $k = 0, 1, \ldots, m$ changing, when β is changing on $[0, \pi)$?
b) How many zeros the n-th eigenfunction $\varphi_n(x, q, \alpha, \beta)$ has, i.e. what is equal m?

By taking $y = \varphi_n(x)$ and $a = x_n^k$ $(k = 0, 1, \ldots, m)$ in (11), we receive

$$\varphi_n'\left(x_n^k\right)\dot{\varphi}_n\left(x_n^k\right) - \dot{\varphi}_n'\left(x_n^k\right)\varphi_n\left(x_n^k\right) - \varphi_n'(0)\,\dot{\varphi}_n(0) + \\ + \dot{\varphi}_n'(0)\,\varphi_n(0) = \int_0^{x_n^k} \varphi_n^2(x)\,dx. \quad (15)$$

Since the initial conditions (13) should be held for all $\mu \in \mathbb{C}$, then $\dot{\varphi}_n(0) = 0$ and $\dot{\varphi}_n'(0) = 0$. Also, taking into account that $\varphi_n\left(x_n^k\right) = 0$, from (15) we get

$$\varphi_n'\left(x_n^k\right)\dot{\varphi}_n\left(x_n^k\right) = \int_0^{x_n^k} \varphi_n^2(x)\,dx. \quad (16)$$

Since the zeros of the solutions are simple, then $\varphi_n'\left(x_n^k\right) \neq 0$, and therefore, from (16) the following equality implies :

$$\frac{\dot{\varphi}_n\left(x_n^k\right)}{\varphi_n'\left(x_n^k\right)} = \frac{1}{\left(\varphi_n'\left(x_n^k\right)\right)^2}\int_0^{x_n^k} \varphi_n^2(x)\,dx. \quad (17)$$

Now, from (7), we obtain

$$\dot{x}_n^k(\mu_n) = \left.\frac{dx_n^k(\mu)}{d\mu}\right|_{\mu=\mu_n} = -\frac{\dot{\varphi}_n\left(x_n^k\right)}{\varphi_n'\left(x_n^k\right)} = -\frac{1}{\left(\varphi_n'\left(x_n^k\right)\right)^2}\int_0^{x_n^k} \varphi_n^2(x)\,dx, \quad (18)$$

i.e. zeros $x_n^k(\mu_n)$, $k = 0, 1, \ldots, m$ of the eigenfunction $\varphi_n(x)$ are decreasing if the eigenvalue $\mu_n(q, \alpha, \beta)$ is increasing, which in its turn means that

$$\dot{x}_n^k(\mu_n(q, \alpha, \beta)) \leq 0. \quad (19)$$

Let us note, that the equality $\dot{x}_n^k(\mu_n) = 0$ may occur only at $x_n^k = 0$, i.e., when $x = 0$ is a zero of the eigenfunction $\varphi_n(x)$, and it is so at $\alpha = \pi$ $(\gamma = \pi l,\ l = 1, 2, \ldots)$.

Meanwhile, in the inequality $\dot{x}_n^k(\mu_n) = \dot{x}_n^k(\mu_n(q, \alpha, \beta)) \leq 0$, the variable μ_n can be changed depending on q, α and β. More precisely, if for some change in these three variables $\mu_n(q, \alpha, \beta)$ increases, then the zeros of the eigenfunction $\varphi_n(x)$ are moving to the left, and if $\mu_n(q, \alpha, \beta)$ decreases, then the zeros of the eigenfunction $\varphi_n(x)$ are moving to the right.

In the work [8], it was proved for $q \in L^2_{\mathbb{R}}[0, \pi]$ that the number of zeros of the n-th eigenfunctions of the problems $L(q, \pi, 0)$ and $L(0, \pi, 0)$ are equal. But it is easy to see that the same proof remains true for $q \in L^1_{\mathbb{R}}[0, \pi]$.

It is easy to calculate that the eigenvalues of the problem $L(0, \pi, 0)$ are $\mu_n(0, \pi, 0) = (n+1)^2$ and the eigenfunctions are

$$\varphi_n(x) = \varphi(x, \mu_n(0, \pi, 0), \pi, 0) = \\ = \varphi\left(x, (n+1)^2, \pi, 0\right) = \frac{\sin(n+1)x}{n+1},\ n = 0, 1, 2, \ldots. \quad (20)$$

The zeros of this eigenfunction are $x_n^k = \dfrac{\pi k}{n+1}$, $k = 0, 1, \ldots, n+1$, i.e. the n-th eigenfunction of the problem $L(0, \pi, 0)$ has $n+2$ zeros in $[0, \pi]$, two of which are 0 and π, and n zeros in $(0, \pi)$.

Thereby, the n-th eigenfunction $\varphi(x, q, \pi, 0)$ of the problem $L(q, \pi, 0)$ has two zeros at the endpoints of $[0, \pi]$, i.e. $x_n^0(q, \pi, 0) = 0$, $x_n^{n+1}(q, \pi, 0) = \pi$ and also n zeros in the interval $(0, \pi)$.

Along with increasing β from 0 to π the eigenvalue $\mu_n(q, \pi, 0)$ is continuously (with respect to β) decreasing from $\mu_n(q, \pi, 0)$ to $\mu_n(q, \pi, \pi) = \mu(\pi, \pi - \pi n) =$ $= \mu(\pi, 0 - (n-1)\pi) = \mu_{n-1}(q, \pi, 0)$ (see [5]) and, according to (18), the zeros of the function $\varphi_n(x, q, \pi, \beta)$ are increasing, i.e. are moving to the right (all but leftmost zero $x_n^0 = 0$). In particular, the rightmost zero $x_n^{n+1} = \pi$, by moving to the right, leaves the segment $[0, \pi]$ and in $[0, \pi]$ remains $n + 1$ zeros (one $x_n^0 = 0$ and n zeros in the interval $(0, \pi)$). And the previous zero reaches π when the relation $\varphi_n(\pi) = \varphi(\pi, \mu_n(q, \pi, \beta), \pi, q) = c_n \psi_n(\pi) \sin\beta = 0$ again occurs (see below (21) and (24)), and this is possible only when β reaches π (and $\mu_n(q, \pi, \beta)$, by decreasing, reaches $\mu_n(q, \pi, \pi) = \mu_{n-1}(q, \pi, 0)$). Then the eigenfunction $\varphi_n(x) = \varphi(x, \mu_n(q, \pi, \beta), \beta, q)$ will smoothly transform to the eigenfunction $\varphi(x, \mu_n(q, \pi, \pi), \pi, q) = \varphi(x, \mu_{n-1}(q, \pi, 0), \pi, q)$ which has $n+1$ zeros in $[0, \pi]$, two out of which are the endpoints 0 and π, and $n - 1$ zeros are in $(0, \pi)$. Thus, the oscillation theorem is proved for all $L(q, \pi, \beta)$, $\beta \in [0, \pi)$.

Now, as $y(x, \mu)$ let us take $\psi(x, \mu, \beta, q) -$ the solution of the equation (1) satisfying the following initial conditions:

$$\psi(\pi, \mu, \beta, q) = \sin\beta, \quad \psi'(\pi, \mu, \beta, q) = -\cos\beta. \tag{21}$$

It is easy to see that the eigenvalues $\mu_n = \mu_n(q, \alpha, \beta)$, $n = 0, 1, \ldots$, of the problem $L(q, \alpha, \beta)$ are the zeros of the entire function

$$\Psi(\mu) = \Psi(\mu, \alpha, \beta, q) = \psi(0, \mu, \beta, q)\cos\alpha + \psi'(0, \mu, \beta, q)\sin\alpha, \tag{22}$$

and the eigenfunctions, corresponding to these eigenvalues, are obtained by the formula

$$\psi_n(x) = \psi(x, \mu_n(q, \alpha, \beta), \beta, q), \ n = 0, 1, \ldots. \tag{23}$$

Since all eigenvalues μ_n are simple, then eigenfunctions $\varphi_n(x)$ and $\psi_n(x)$ corresponding to the same eigenvalue μ_n are linearly dependent, i.e. there exist the constants $c_n = c_n(q, \alpha, \beta)$, $n = 0, 1, \ldots$, such that

$$\varphi_n(x) = c_n \psi_n(x), \ n = 0, 1, \ldots. \tag{24}$$

This implies that $\varphi_n(x, q, \alpha, \beta)$ and $\psi_n(x, q, \alpha, \beta)$ have equal number of zeros.

Let $0 \le x_n^m < x_n^{m-1} < \cdots < x_n^0 \le \pi$ be the zeros of the eigenfunction $\psi_n(x, q, \alpha, \beta)$ i.e. $\psi_n\left(x_n^k, q, \alpha, \beta\right) = 0$, $k = 0, 1, \ldots, m$.

By taking in the identity (12) $y = \psi_n(x) = \psi(x, \mu_n(q, \alpha, \beta), \beta, q)$, we receive

$$\begin{gathered}\psi_n'(\pi)\dot{\psi}_n(\pi) - \dot{\psi}_n'(\pi)\psi_n(\pi) - \psi_n'\left(x_n^k\right)\dot{\psi}_n\left(x_n^k\right) + \\ + \dot{\psi}_n'\left(x_n^k\right)\psi_n\left(x_n^k\right) = \int_{x_n^k}^{\pi} \psi_n^2(x)\,dx.\end{gathered} \tag{25}$$

From (21) we get that $\dot{\psi}_n(\pi) = 0$ and $\dot{\psi}_n'(\pi) = 0$. And since $\psi_n\left(x_n^k\right) = 0$, then the equality (25) gets the form

$$-\psi_n'\left(x_n^k\right)\dot{\psi}_n\left(x_n^k\right) = \int_{x_n^k}^{\pi} \psi_n^2(x)\,dx. \tag{26}$$

As all zeros x_n^k are simple, i.e. $\psi_n'\left(x_n^k\right) \neq 0$, dividing both sides of the last equality by $\left(\psi_n'\left(x_n^k\right)\right)^2$, we obtain

$$\frac{\dot{\psi}_n\left(x_n^k\right)}{\psi_n'\left(x_n^k\right)} = -\frac{1}{\left(\psi_n'\left(x_n^k\right)\right)^2} \int_{x_n^k}^{\pi} \psi_n^2(x)\, dx. \tag{27}$$

Now from (7), by taking $y = \psi_n(x)$, $x = x_n^k$, we have

$$\dot{x}_n^k(\mu_n) = \left.\frac{dx_n^k(\mu)}{d\mu}\right|_{\mu=\mu_n} = -\frac{\dot{\psi}_n\left(x_n^k\right)}{\psi_n'\left(x_n^k\right)} = \frac{1}{\left(\psi_n'\left(x_n^k\right)\right)^2} \int_{x_n^k}^{\pi} \psi_n^2(x)\, dx \geq 0, \tag{28}$$

i.e. the zeros $x_n^k(\mu_n)$, $k = 0, 1, \ldots, m$ of the eigenfunction $\psi_n(x)$ are increasing, if the eigenvalue μ_n is increasing. Note that the equality $\dot{x}_n^k(\mu_n) = 0$ is possible only when $x_n^0(\mu_n) = \pi$, but it holds when $\beta = 0$ $(\delta = -\pi l,\ l = 0, 1, 2, \ldots)$.

While studying the dependence of the zeros of eigenfunctions on α, it is convenient to use formula (28), because the eigenfunctions $\psi_n(x)$ have fixed values $\psi_n(\pi, \mu, \beta) = \sin\beta$ and $\psi_n'(\pi, \mu, \beta) = -\cos\beta$ for all $\mu \in \mathbb{C}$, i.e. all $\psi_n(x)$ satisfy the initial conditions $\psi_n(\pi) = \sin\beta$ and $\psi_n'(\pi) = -\cos\beta$, which means that from the endpoint π of the segment $[0, \pi]$ (when changing α) new zeros can neither enter nor leave (neither appear nor disappear). Thus, with increasing α, the eigenvalues $\mu_n(q, \alpha, \beta)$ (with fixed q and β) are increasing and according to (28) (i.e. $\dot{x}_n^k(\mu_n) \geq 0$) the zeros of the eigenfunction $\psi_n(x)$ are moving to the right (i.e. are increasing). Wherein, the values $\psi(\pi) = \sin\beta$ and $\psi'(\pi) = -\cos\beta$ are not changed, the number of the zeros increase, and these zeros can neither "collide" nor "split" as they are simple. That's why new zeros can appear only entering through $0-$ left endpoint of the segment $[0, \pi]$ and moving to the right (and "condensing" respectively). And new zeros are entered through $0-$ left endpoint of the segment $[0, \pi]$ only when $\psi_n(0) = 0$, but since $\psi_n(0) = c_n \varphi_n(0) = c_n \sin\alpha$ $(c_n \neq 0)$, then the equality $\psi_n(0) = 0$ is possible only when $\sin\alpha = 0$, i.e. in our notations only when $\alpha = \pi$ (and because $\mu_n(q, 0, \beta) = \mu(0 + \pi n, \beta) = \mu(\pi + (n-1)\pi, \beta) = \mu_{n-1}(q, \pi, \beta)$) or when $\alpha = 0$.

Hence, when $\alpha = \pi$, the eigenfunction $\psi_n(x, q, \pi, \beta)$ as well as $\varphi_n(x, q, \pi, \beta)$ has n zeros in $(0, \pi)$ and one zero at $x = 0$ left endpoint. Wherein, $\psi_n(x, q, \pi, \beta) = \psi(x, \mu_n(q, \pi, \beta), \beta, q) = \psi(x, \mu_{n+1}(q, 0, \beta), \beta, q) = \psi_{n+1}(x, q, 0, \beta)$. With increasing α from 0 to π the eigenvalue $\mu_{n+1}(q, 0, \beta)$ is increasing (continuously with respect to α) to $\mu_{n+1}(q, \pi, \beta)$, and the leftmost zero $x = 0$ by moving to the right, appears in $(0, \pi)$, i.e. there are $n+1$ zeros of the eigenfunction $\psi_n(x, q, \alpha, \beta)$ in $(0, \pi)$ (and another fixed zero $x = \pi$, if $\beta = 0$). A new zero will appear at $x = 0$ left endpoint, when α will reach $\alpha = \pi$.

Thus, we obtain the following oscillation theorem:

Theorem 1. *The eigenfunctions of the problem $L(q, \alpha, \beta)$ corresponding to the n-th eigenvalue $\mu_n(q, \alpha, \beta)$, $n = 0, 1, 2, \ldots$, have exactly n zeros in $(0, \pi)$. All these zeros are simple. If $\alpha = \pi$ and $\beta = 0$, then the n-th eigenfunction has $n + 2$ zeros in $[0, \pi]$, and if either $\alpha = \pi$, $\beta \in (0, \pi)$ or $\beta = 0$, $\alpha \in (0, \pi)$, then the n-th eigenfunction has $n + 1$ zeros in $[0, \pi]$.*

Oscillation properties of the solutions of the problem $L(q, \alpha, \beta)$ (Sturm theory), the studies of which initiated by Sturm in [9, 10], outlined in monographic literature (see, e.g. [1, 11, 12]) for continuous q. In the recent years, the study was mostly focused on the cases when q is bounded or $q \in L^2_{\mathbb{R}}[0, \pi]$ (see, e.g. [8, 13]), but in many studies (see [14] and references therein) implicitly assumes that Sturm's oscillation theorem (that the n-th eigenfunction has n zeros) is also true for $q \in L^1_{\mathbb{R}}[0,\pi]$, although the rigorous proof is not available in the literature.

Our oscillation theorem is true for all $q \in L^1_{\mathbb{R}}[0,\pi]$.

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Tigran Harutyunyan

Faculty of Mathematics and Mechanics, Yerevan State University, 1 Alex Manoogian, 0025, Yerevan, Armenia

E-mail address: hartigr@yahoo.co.uk

Avetik Pahlevanyan

Faculty of Mathematics and Mechanics, Yerevan State University, 1 Alex Manoogian, 0025, Yerevan, Armenia

E-mail address: avetikpahlevanyan@gmail.com

Yuri Ashrafyan

Faculty of Mathematics and Mechanics, Yerevan State University, 1 Alex Manoogian, 0025, Yerevan, Armenia

E-mail address: yuriashrafyan@ysu.am