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COMPLEX ANALYSIS

On Extreme Points of Some Sets of Holomorphic Functions*

A. I. Petrosyan[1]*

[1] *Yerevan State University*

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Abstract—In the Banach space $H^\infty(U)$ of bounded, holomorphic functions in the unit disc U of the complex plane, the paper considers the subset $H^\infty(U;K)$ of functions that take values in a compact K. A description of the extreme points of $H^\infty(U;K)$ is given under rather general conditions on K.

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1. Let B be a Banach space, $X \subset B$ be some convex subset. A point x is called extreme for the set X if it is not a proper convex combination of two different points from X. Equivalently, this means that if $x + t \in X$ and $x - t \in X$ for an element $t \in X$, then $t = 0$.

There is much work devoted to the problem of description of the extreme points of convex sets. In many cases the extreme points contain some significant information on the geometry of the set. For instance, any closed, bounded, convex set is a convex hull of the set of its own extreme points in the finite-dimensional space. Note that this statement is not true in general Banach space. For instance, the unit ball of the space l_1 has no extreme points. But if B is conjugate to some Banach space, then Krein–Milman theorem (see, eg. [1]) implies that the unit ball of B not only has some extreme points, but the set of these points can be broad enough: the ball is a weak closure of the convex hull of its extreme points.

This paper considers the case where B is a space of bounded, holomorphic functions.

2. Below by U we denote the unit disc of the complex plane $\mathbb{C}$, i.e. $U = \{z \in \mathbb{C}\colon\ |z| < 1\}$, by K a convex compact in the w-plane, by $H^\infty(U)$ the space of bounded, holomorphic functions in U and by $H^\infty(U;K)$ the set of functions $f(z) \in H^\infty(U)$, for which $f(U) \subset K$. Note that $H^\infty(U;K)$ is a convex set in $H^\infty(U)$.

If K is the disc $|w| \le 1$, then $H^\infty(U;K)$ is the unit ball in $H^\infty(U)$, and its extreme points (see [2], p. 197) are those and only those functions f that satisfy

$$\int_0^{2\pi} \log\left(1 - \left|f\left(e^{i\theta}\right)\right|\right) d\theta = -\infty. \tag{1}$$

The present paper is aimed at description of the extreme points of the set $H^\infty(U;K)$ for a sufficiently wide class of convex compacts K, namely, for compacts satisfying the condition

(P) *there exists a natural number n such that at any point $\zeta \in \partial K$ there exist a straight line l_ζ supporting K and a parabola of the order $2n$, with vertex at ζ, tangential to l_ζ, such that in some neighborhood D_ζ of the point ζ the branches of the parabola envelope the set $K \cap D_\zeta$.*

*E-mail: albpet@xter.net

Note that the (P) means that the tangency order of the boundary ∂K with the support line does not exceed $2n-1$ at any point of ∂K. Recall that a support line to a given set K at a given point $\zeta \in \partial K$ is the straight line passing trough ζ and having K in one of its sides.

The paper [3] gives a description of the extreme points of the set $H^\infty(U;K)$ for compacts satisfying a condition easily proved to be equivalent to (P) for $n=1$.

We define the following nonnegative function on the set K:

$$G(w) = \sup\left\{|\zeta|:\ w \pm \zeta \in K\right\}, \quad w \in K, \tag{2}$$

and by $\rho(w, \partial K)$ denote the distance between w and ∂K, i.e.

$$\rho(w, \partial K) = \inf_{t \in \partial K} |w - t|.$$

Then the following statement is true.

Lemma 1. *For a convex compact K satisfying (P)*

$$G(w) \le C\left\{\rho(w, \partial K)\right\}^{\frac{1}{n}}, \qquad w \in K, \tag{3}$$

where C is independent of w.

Proof: The neighborhoods D_ζ mentioned in (P), totally cover the boundary of the compact K. Besides, the set

$$K' = K \setminus \left(\bigcup_{\zeta \in \partial K} D_\zeta\right)$$

is a compact inside K, i.e. $\rho(K', \partial K) > 0$. Further, the compactness of K and (2) imply that the function $G(w)$ is bounded by some constant: $G(w) \le M$.

Suppose now, that

$$C = \max\left\{1;\ M\left[\rho\left(K', \partial K\right)\right]^{-1/n}\right\}.$$

Then obviously

$$G(w) \le M \le C\left[\rho\left(K', \partial K\right)\right]^{1/n} \le C\left[\rho\left(w, \partial K\right)\right]^{1/n} \tag{4}$$

for $w \in K'$. Besides, if $w \in K \setminus K'$, then we choose $\zeta_0 \in \partial K$ to have

$$\rho(w, \partial K) = \rho(w, \zeta_0). \tag{5}$$

Observe that the support line to the compact K through the point ζ_0 obviously is perpendicular to the line d connecting the points w and ζ_0, otherwise there would be some points on ∂K closer to w than ζ_0, which contradicts (5). For the corresponding parabola of the degree n, with the axis d and the vertex ζ_0,

$$G(w) \le |w - \zeta|^{1/n} = \left[\rho\left(w, \partial K\right)\right]^{1/n}. \tag{6}$$

By (4) and (6), we come to (3) for all points $w \in K$.

Theorem 1. *Let K be a convex compact. If a function $f(z)$ is an extreme point of the set $H^\infty(U,K)$, then necessarily K satisfies*

$$\int_0^{2\pi} \log \rho\left(f\left(e^{i\theta}\right), \partial K\right) d\theta = -\infty, \tag{7}$$

and conversely if K satisfies the condition (P) and (7), then $f(z)$ is an extreme point of the set $H^\infty(U,K)$.

Proof: We prove the necessity by contradiction, i.e. we suppose that the integral (7) converges. Then the function

$$h(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\rho\left(f\left(e^{i\theta}\right),\partial K\right)d\theta\right\} \tag{8}$$

is holomorphic and bounded in the disc U and $|h(e^{i\theta})| = \rho\big(f(e^{i\theta}),\partial K\big)$ for almost all θ. Hence $f(e^{i\theta}) \pm h(e^{i\theta}) \in K$ almost everywhere, and therefore $f(z) \pm h(z) \in K$, $z \in U$. Besides, $h(z) \not\equiv 0$, and consequently $f(z)$ is not an extreme point.

In fact [4] considers the case where K is an arbitrary convex set different from a half-plane and the whole complex plane $\mathbb{C}$ and gives a necessary condition

$$\int_0^{2\pi}\log\frac{\rho\left(f\left(e^{i\theta}\right),\partial K\right)}{\rho\left(f\left(e^{i\theta}\right),\partial K\right)+1}d\theta = -\infty. \tag{9}$$

The conditions (7) and (9) are equivalent when K is bounded.

Turning to sufficiency, suppose $g(z)$ is holomorphic in U and $f(z) \pm g(z) \in K$. Then $|g(z)| \le G\left(f(z)\right)$ by the definition (2) of $G(w)$. By (3)

$$\left|g\big(e^{i\theta}\big)\right| \le G\left(f\big(e^{i\theta}\big)\right) \le C\left[\rho\left(f\big(e^{i\theta}\big),\partial K\right)\right]^{1/n}.$$

Taking logarithms and integrating, by (7) we get

$$\int_0^{2\pi}\log\left|g\big(e^{i\theta}\big)\right|d\theta \le \log C + \frac{1}{n}\int_0^{2\pi}\log\rho\left(f\big(e^{i\theta}\big),\partial K\right)d\theta = -\infty.$$

Hence we conclude that $g(z) \equiv 0$, i.e. $f(z)$ is an extreme point of the set $H^\infty(U;K)$. The proof is complete.

If K is a disc, then $\rho\big(f(e^{i\theta}),\partial K\big) = 1 - |f(e^{i\theta})|$ and (7) reduces to (1). In [4], the sufficiency of the condition (7) is proved in the case where the boundary of the compact K is a twice smooth curve with positive curvature in all points. This implies that at all points of that curve the tangency is not more than of the first order and consequently K satisfies (P). Thus, the mentioned result of [4] follows from Theorem 1.

3. A counterpart of Theorem 1 is valid for the Banach space $A(U)$ of functions analytic in the disc and continuous up to its boundary. Namely, if by $A(U;K)$ we denote the set of all those functions of $A(U)$, which take vales in K, then the following statement is true.

Theorem 2. *Let K be a convex compact. Then the condition* (7) *is necessary and, for compacts satisfying* (P), *also sufficient for the function $f(z)$ to be an extreme point of the set $A(U;K)$.*

Proof: The sufficiency is proved in the same way as in Theorem 1. As for the necessity, we have to provide the continuity of the boundary values of the function $h(z)$ of (8). If $f \in A(U;K)$ and $\log\rho\big(f(e^{i\theta}),\partial K\big)$ is integrable, then we can choose a function u continuous on the unit circle, so that $0 \le u \le \rho\big(f(e^{i\theta}),\partial K\big)$, $\log u$ be integrable and u be continuously differentiable on any open arc of the set where $f \notin \partial K$. The function

$$h(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log u\,d\theta\right\},$$

has continuous boundary values and satisfies the condition $\left|h(e^{i\theta})\right| \le \rho\big(f(e^{i\theta}),\partial K\big)$. Consequently, $f(e^{i\theta}) \pm h(e^{i\theta}) \in K$ for all θ, and hence $f(z) \pm h(z) \in K$, $z \in U$. Besides, $h(z) \not\equiv 0$, and therefore $f(z)$ is not an extreme point of the set $A(U;K)$. This completes the proof.

The case when K is a convex polyhedron was studied in [3]. In that case, there are essentially less extreme points of $A(U;K)$: they are those and only those functions $f \in A(U;K)$ $(f \ne \text{const})$ for which $f(e^{i\theta}) \subset \partial K$ or, in other words, $\rho\big(f(e^{i\theta}),\partial K\big) \equiv 0$.

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