On weighted spaces of functions harmonic in $\mathbb{R}^n$

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Abstract. The paper establishes integral representation formulas in arbitrarily wide Banach spaces $b^p_\omega(\mathbb{R}^n)$ of functions harmonic in the whole $\mathbb{R}^n$.

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1. Introduction

1.1. This paper extends the results of [1] related to arbitrarily wide spaces of functions harmonic in the unit ball B of $\mathbb{R}^n$ to similar spaces of functions harmonic in the whole $\mathbb{R}^n$. Namely, the integral representation formulas in spaces $b^p_\omega(\mathbb{R}^n)$, which have natural definition, are obtained by exhausting $\mathbb{R}^n$ by enlarging balls. Also, a representation connected with the natural isometry between $b^2_\omega(\mathbb{R}^n)$ and the space $L^2(S)$ is obtained, which is of an explicit form of integral operator along with its inversion.

The results of this paper are faregoing multidimensional similarities of the early results of M.M. Djrbashian [3], [4] (1945–1948) which in essence gave rise to the theory of $H^p(\alpha)$ spaces in the unit disc: the applied analytic apparatus allows to extend the results of [5] related to the one-dimensional case and holomorphic functions to functions harmonic in $\mathbb{R}^n$.

Note that in [2], the case of weighted spaces of functions analytical in the unit ball of $\mathbb{C}^n$ is investigated.

1.2. We start with some notation which we use all over the paper.

$B = \{x \in \mathbb{R}^n\colon\ |x| < 1\}$ is the open unit ball in $\mathbb{R}^n$ and $S = \{x \in \mathbb{R}^n\colon\ |x| = 1\}$ is its boundary, i.e. S is the unit sphere in $\mathbb{R}^n$;

σ is the normalized surface-area measure over S, i.e. $\sigma(S) = 1$;

$\mathcal{H}_k(\mathbb{R}^n)$ is the set of all complex-valued homogeneous harmonic polynomials of degree k in $\mathbb{R}^n$;

$\mathcal{H}_k(S)$ is the set of all spherical harmonics of degree k, i.e. all restrictions of functions from $\mathcal{H}(\mathbb{R}^n)$ to the sphere S;

$Z_k(\eta,\zeta)$ is the zonal harmonic of degree k, i.e. $Z_k(\cdot,\zeta) \in \mathcal{H}_k(S)$ and $p(\zeta) = \int_S p(\eta) Z_k(\eta,\zeta)\, d\sigma(\eta)$ for all $p \in \mathcal{H}_k(S)$;

$P[f]$ is used for the Poisson integral of f:

$$P[f](x) = \int_S P(x,\zeta) f(\zeta)\, d\sigma(\zeta), \quad \text{where} \quad P(x,\zeta) = \frac{1-|x|^2}{|\zeta - x|^n}. \tag{1}$$

2. The case of the ball

We shall use the following definitions and statements from [1] related to the weighted spaces $b^p_\omega(B)$ in the unit ball.

As in [5], by Ω we denote the class of all functions $\omega(t)$ defined on $[0,1]$ and such that $\omega(1) = \omega(1-0)$ and

(i) $0 < \bigvee_\delta^1 \omega < \infty$ for any $\delta \in [0,1)$;

(ii) $\Delta_k \equiv \Delta_k(\omega) = -\int_0^1 t^k\, d\omega(t) \neq 0, \infty, \quad k = 0, 1, \ldots;$

(iii) $\liminf_{k\to\infty} \sqrt[k]{|\Delta_k|} \geq 1$.

Further, for a given $\omega \in \Omega$, we denote

$$d\mu_\omega(x) = -d\omega(r^2)\, d\sigma(\zeta),$$

where $x = r\zeta$ is the polar form of x, (i.e. $r = |x|$, $\zeta \in S$) and define $L^p_\omega(B)$ as the set of all $d\mu_\omega$-measurable functions in B for which

$$\|u\|_{p,\omega} = \left\{ \int_B |u(x)|^p\, |d\mu_\omega(x)| \right\}^{1/p} < +\infty, \qquad 1 \leq p < \infty.$$

By $b^p_\omega(B)$ we denote the subset of harmonic functions from $L^p_\omega(B)$.

Further, for a given $\omega \in \Omega$ we use the ω-kernel of the form

$$R_\omega(x,y) = \sum_{k=0}^{\infty} \Delta_k^{-1} Z_k(x,y),$$

where $Z_k(x,y)$ is the harmonic extension of the zonal harmonic Z_k by its both arguments. As it is proved in [1], for any function $u \in b^p_\omega(B)$ the following integral representation is true:

$$u(x) = \int_B u(y) R_\omega(x,y)\, d\mu_\omega(y), \quad x \in B. \tag{2}$$

3. The integral representation in $\mathbb{R}^n$

3.1. Let Ω^∞ denote the set of parameter-functions $\omega(t)$, which strictly decrease on the whole half-axis $[0,+\infty)$ and are such that $\omega(0)=1$ and

$$\Delta_k^\infty(\omega) = -\int_0^{+\infty} t^k\, d\omega(t) < +\infty \text{ for any } \; k=0,1,\dots \; .$$

For a given $\omega \in \Omega^\infty$ we introduce the space $b_\omega^p(\mathbb{R}^n)$ as the set of all functions which are harmonic in $\mathbb{R}^n$ and such that

$$\|u\|_{p,\omega} = \left\{ \int_{\mathbb{R}^n} |u(y)|^p\, d\mu_\omega(y) \right\}^{1/p} < +\infty, \quad 1 \le p < +\infty,$$

where $d\mu_\omega(r\zeta) = -d\omega(r^2)d\sigma(\zeta)$. Let $L_\omega^p(\mathbb{R}^n)$ be the corresponding Lebesgue spaces. We shall deal with the following ω-kernel in $\mathbb{R}^n$:

$$R_\omega^\infty(x,y) = \sum_{k=0}^\infty \frac{Z_k(x,y)}{\Delta_k^\infty(\omega)} \,. \tag{3}$$

Lemma 1. *The right-hand side series in* (3) *is absolutely and uniformly convergent on any compact subset of* $\mathbb{R}^n \times \mathbb{R}^n$*, and hence* $R_\omega^\infty(x,y)$ *is harmonic in each of its variables in* $\mathbb{R}^n$*.*

Proof: Let $x=r\zeta$, $y=\rho\eta$, where $\zeta,\eta \in S$. As the function $Z_k(x,y)$ is homogeneous in its both variables, we get

$$|Z_k(x,y)| = r^k\rho^k|Z_k(\zeta,\eta)| \le r^k\rho^k h_k, \tag{4}$$

where h_k is the dimension of $\mathcal{H}_k(S)$. Now observe that under the above conditions

$$\lim_{k\to\infty} \sqrt[k]{\Delta_k^r(\omega)} = r^2 \quad \text{for} \quad \Delta_k^r(\omega) = -\int_0^{r^2} t^k\, d\omega(t) \quad \text{and} \quad \forall r \in (0,+\infty]. \tag{5}$$

Indeed, it is obvious that $\Delta_k^r(\omega) \le r^{2k}\bigl(1-\omega(r^2)\bigr)$ and hence

$$\limsup_{k\to\infty} \sqrt[k]{\Delta_k^r(\omega)} \le r^2. \tag{6}$$

On the other hand,

$$\Delta_k^r(\omega) \ge -\int_\delta^{r^2} t^k\, d\omega(t) \ge \delta^k\bigl(\omega(r^2)-\omega(\delta)\bigr)$$

for any $\delta \in (0, r^2)$. Therefore

$$\liminf_{k\to\infty} \sqrt[k]{\Delta_k^r(\omega)} \geq \delta \lim_{k\to\infty} \sqrt[k]{\left(\omega(r^2) - \omega(\delta)\right)} = \delta,$$

and hence by (6) the passage $\delta \to r^2$ gives (5). Further, note that $\Delta_k^r(\omega) \uparrow_r$. Therefore by (5)

$$\liminf_{k\to\infty} \sqrt[k]{\Delta_k^\infty(\omega)} \geq r^2$$

for any $r > 0$, and consequently

$$\lim_{k\to\infty} \sqrt[k]{\Delta_k^\infty(\omega)} = +\infty. \tag{7}$$

The desired convergence follows from (4) and (7) in view of the estimate

$$h_k \leq C k^{n-2} \tag{8}$$

(see, e.g. [7]). □

3.2. The following statement is the main theorem of this section.

Theorem 1. *Let $u \in b_\omega^2(\mathbb{R}^n)$, where $\omega \in \Omega^\infty$. Then*

$$u(x) = \int_{\mathbb{R}^n} u(y) R_\omega^\infty(x,y)\, d\mu_\omega(y), \quad x \in \mathbb{R}^n. \tag{9}$$

Proof: The idea of the proof is the following. For any $r > 0$ we introduce a kernel

$$R_\omega^r(x,y) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{Z_k(x,y)}{\Delta_k^r(\omega)},$$

where $\Delta_k^r(\omega)$ is defined in (5). This kernel plays for a ball $|x| < r$ the same role, as $R_\omega(x,y)$ for a unit ball, after dilation we obtain the integral representation (10) from (2). And passing to limits as $r \to \infty$ we get (9) from (10), which is expected, because the coefficients $\Delta_k^r(\omega)$ of the expansion of the kernel $R_\omega^r(x,y)$ tend to the coefficients $\Delta_k^\infty(\omega)$ of $R_\omega^\infty(x,y)$.

Consider the function $\omega_r(t) = \omega(r^2 t)$, $0 \leq t \leq 1$. Then obviously $\Delta_k(\omega_r) = r^{-2k}\Delta_k^r(\omega)$. Therefore by (5)

$$\lim_{k\to\infty} \sqrt[k]{\Delta_k(\omega_r)} = 1,$$

and hence $\omega_r \in \Omega$. On the other hand, $u(rx) \in b^2_{\omega_r}(\mathbf{B}^n)$. Thus, the representation (2) is valid for $u(rx)$. Now observe that for $|x| < r$, $|y| < r$

$$R_{\omega_r}\left(\frac{x}{r}, \frac{y}{r}\right) \equiv \sum_{k=0}^{\infty} \frac{Z_k(x,y)}{r^2 \Delta_k(\omega)} = \sum_{k=0}^{\infty} \frac{Z_k(x,y)}{\Delta_k^r(\omega)} \stackrel{\text{def}}{=} R_\omega^r(x,y),$$

and $d\mu_{\omega_r}\left(\frac{y}{r}\right) = d\mu_\omega(y)$. Consequently, (2) can be written in the form

$$u(x) = \int_{|y|<r} u(y) R_\omega^r(x,y)\, d\mu_\omega(y), \quad |x| < r, \tag{10}$$

and to prove (9) it suffices to show that for any fixed $x \in \mathbb{R}^n$

$$\lim_{r\to\infty} \int_{|y|<r} u(y) R_\omega^r(x,y)\, d\mu_\omega(y) = \int_{\mathbb{R}^n} u(y) R_\omega^\infty(x,y)\, d\mu_\omega(y). \tag{11}$$

To prove this relation, observe that by Lemma 1 the function $R_\omega^\infty(x,\cdot)$ is harmonic in $\mathbb{R}^n$. Therefore, by Hölder's inequality

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|u(y) R_\omega^\infty(x,y)\right| d\mu_\omega(y) &\le \|u\|_{2,\omega} \left\{ \int_{\mathbb{R}^n} \left|R_\omega^\infty(x,y)\right|^2 d\mu_\omega(y) \right\}^{1/2} \\
= \|u\|_{2,\omega} &\left\{ \sum_{k=0}^{\infty} \frac{1}{(\Delta_k^\infty)^2} \int_{\mathbb{R}^n} Z_k^2(x,\rho\zeta)\, d\mu_\omega(\rho\zeta)| \right\}^{1/2} \\
= \|u\|_{2,\omega} &\left\{ \sum_{k=0}^{\infty} \frac{1}{(\Delta_k^\infty)^2} \int_0^\infty \rho^{2k}\, |d\omega(\rho^2)| \int_S Z_k^2(x,\zeta)\, d\sigma(\zeta) \right\}^{1/2}.
\end{aligned}$$

Further, it is evident that

$$\int_S Z_k^2(x,\zeta)\, d\sigma(\zeta) = |x|^{2k} \int_S Z_k^2\left(\frac{x}{|x|}, \zeta\right) d\sigma(\zeta) = |x|^{2k} h_k. \tag{12}$$

Consequently,

$$\int_{\mathbb{R}^n} \left|u(y) R_\omega^\infty(x,y)\right| d\mu_\omega(y) \le \|u\|_{2,\omega} \left\{ \sum_{k=0}^{\infty} \frac{|x|^{2k} h_k}{\Delta_k^\infty} \right\}^{1/2}. \tag{13}$$

According to (7) and (8), the right-hand side a series of this estimate is convergent. Hence we conclude that the relation (11) is equivalent to

$$\lim_{r\to\infty} \int_{|y|<r} u(y) [R_\omega^r(x,y) - R_\omega^\infty(x,y)]\, d\mu_\omega(y) = 0.$$

In order to prove the latter relation, assume that $|x| = r_0$, $r_0+1 < r_1 < r < +\infty$, and observe that

$$\begin{aligned} I(r) &\equiv \left| \int_{|y|<r} u(y)(R^r_\omega(x,y) - R^\infty_\omega(x,y))\, d\mu_\omega(y) \right| \\ &\le \int_{|y|<r_1} |u(y)(R^r_\omega(x,y) - R^\infty_\omega(x,y))|\, d\mu_\omega(y) \\ &\quad + \int_{r_1<|y|<r} |u(y) R^r_\omega(x,y)|\, d\mu_\omega(y) \\ &\quad + \int_{r_1<|y|<r} |u(y) R^\infty_\omega(x,y)|\, d\mu_\omega(y) \equiv I_1(r) + I_2(r) + I_3(r). \end{aligned}$$

To estimate the summand $I_2(r)$ we once again apply Hölder's inequality:

$$\begin{aligned} I_2(r) &\le \left\{ \int\limits_{r_1<|y|<r} |u(y)|^2\, d\mu_\omega(y) \int\limits_{r_1<|y|<r} |R^r_\omega(x,y)|^2\, d\mu_\omega(y) \right\}^{1/2} \\ &= \left\{ \int\limits_{r_1<|y|<r} |u(y)|^2\, d\mu_\omega(y) \sum_{k=0}^\infty \frac{1}{(\Delta^r_k)^2} \int\limits_{r_1<|y|<r} Z^2_k(x,y)\, d\mu_\omega(y) \right\}^{1/2}. \end{aligned} \tag{14}$$

Further, according to (12)

$$\begin{aligned} \int\limits_{r_1<|y|<r} Z^2_k(x,y)\, d\mu_\omega(y)| &= \int_S Z^2_k(x,y)\, d\sigma(\zeta) \int_{r_1}^r \rho^{2k}\, |d\omega(\rho^2)| \\ &= |x|^{2k} h_k \int_{r_1}^r \rho^{2k}\, |d\omega(\rho^2)|. \end{aligned} \tag{15}$$

Therefore

$$\begin{aligned} &\sum_{k=0}^\infty \frac{1}{(\Delta^r_k)^2} \int_{r_1<|y|<r} Z^2_k(x,\rho\eta)\, d\mu_\omega(\rho\eta) \\ &\le \sum_{k=0}^\infty \frac{|x|^{2k}}{(\Delta^r_k)^2} \int_{r_1}^r \rho^{2k} |\, d\omega(\rho^2)| \int_S Z^2_k\left(\frac{x}{|x|}, \eta\right)\, d\sigma(\eta) \le \sum_{k=0}^\infty \frac{|x|^{2k} h_k}{\Delta^r_k} \le \sum_{k=0}^\infty \frac{r_0^{2k} h_k}{\Delta^{r_0+1}_k}. \end{aligned}$$

The last series converges in virtue of (5) and (8), therefore $I_2(r) < \varepsilon/3$ for a given $\varepsilon > 0$ and r_1 large enough. On the other hand, from (13) it follows that $I_3(r) < \varepsilon/3$ for r_1 large enough. Besides, for any fixed r_1

$$\begin{aligned} I_1(r) &\le \|u\|_{2,\omega} \left\{ \int_{|y|<r_1} |R^r_\omega(x,y) - R^\infty_\omega(x,y)|^2\, d\mu_\omega(y) \right\}^{1/2} \\ &= \|u\|_{2,\omega} \left\{ \sum_{k=0}^\infty \left(\frac{1}{\Delta^r_k} - \frac{1}{\Delta^\infty_k} \right)^2 \int_{|y|<r_1} Z^2_k(x,y)\, d\mu_\omega(y) \right\}^{1/2}. \end{aligned} \tag{16}$$

Therefore, by (15)

$$I_1(r) \le \|u\|_{2,\omega} \left\{ \sum_{k=0}^{\infty} \left(\frac{1}{\Delta_k^r} - \frac{1}{\Delta_k^\infty} \right)^2 |x|^{2k} h_k \int_0^{r_1} \rho^{2k} \, |d\omega(\rho^2)| \right\}^{1/2},$$

and the latter series has a convergent majorant independent of r. Indeed,

$$\begin{aligned} &\sum_{k=0}^{\infty} \left(\frac{1}{\Delta_k^r} - \frac{1}{\Delta_k^\infty} \right)^2 |x|^{2k} h_k \int_0^{r_1} \rho^{2k} \, |d\omega(\rho^2)| \\ &\quad \le \sum_{k=0}^{\infty} |x|^{2k} h_k \left(\frac{2}{\Delta_k^r} \right)^2 \Delta_k^{r_1} \le 4 \sum_{k=0}^{\infty} \frac{r_0^{2k} h_k}{\Delta_k^{r_1}} < +\infty, \end{aligned}$$

where the right-hand side series converges due to (5) and (8), as $r_0 < r_1$. Therefore, the right-hand side of (16) vanishes as $r \to +\infty$ and hence $I_1(r) < \varepsilon/3$ for r large enough. Thus, we conclude that $I(r) \to 0$ as $r \to +\infty$, which implies the desired representation (9). □

3.3. As $b^2_\omega(\mathbb{R}^n)$ is a closed subspace of the Hilbert space $L^2_\omega(\mathbb{R}^n)$, there is a unique orthogonal projection Q of $L^2_\omega(\mathbb{R}^n)$ onto $b^2_\omega(\mathbb{R}^n)$, which is described by

Theorem 2. *The operator*

$$Q_\omega[u](x) = \int_{\mathbb{R}^n} u(y) R^\infty_\omega(x,y) \, d\mu_\omega(y), \quad x \in \mathbb{R}^n, \quad u \in L^2_\omega(\mathbb{R}^n),$$

is the orthogonal projection of $L^2_\omega(\mathbb{R}^n)$ onto $b^2_\omega(\mathbb{R}^n)$.

The proof of this theorem as well as of the statements below follows the same lines as the corresponding one in [1] and is thus omitted.

Proposition 1. *Let a function $\widetilde{\omega} \in \Omega^\infty$ be continuously differentiable in $[0,+\infty)$ and such that $\widetilde{\omega}(+\infty) = 0$, $\widetilde{\omega}'(t) < 0$ and is bounded on $[0,+\infty)$ and $\int_0^{+\infty} t^{-1} \, d\widetilde{\omega}(t) > -\infty$. Further, let ω be the Volterra square of $\tilde{\omega}$, i.e.*

$$\omega(x) = -\int_0^\infty \tilde{\omega}\left(\frac{x}{t}\right) d\tilde{\omega}(t), \qquad 0 < x < 1. \tag{17}$$

Then $\omega \in \Omega^\infty$ and

$$\Delta^\infty_m(\omega) = \left[\Delta^\infty_m(\tilde{\omega})\right]^2, \quad m \ge 0. \tag{18}$$

By $h^p(B)$ we denote the ordinary harmonic Hardy space in B. Besides, we consider the operator

$$L_{\tilde{\omega}}[u](x) = -\int_0^\infty u(tx) \, d\tilde{\omega}(t).$$

The following two theorems establish an isometry along with its inversion between $b^2_\omega(\mathbb{R}^n)$ and $L^2(S)$.

Theorem 3. *The mapping $f \mapsto R_{\tilde{\omega}}[f]$, where*

$$R_{\tilde{\omega}}[f](x) = \int_S f(\zeta) R_{\tilde{\omega}}^{\infty}(x,\zeta)\, d\sigma(\zeta)$$

is a linear isometry from $L^2(S)$ to $b_{\omega}^2(\mathbb{R}^n)$.

Theorem 4. *Let $f \in L^2(S)$ and $u = R_{\tilde{\omega}}[f]$. Then*

(a) *$L_{\tilde{\omega}}[u] = P[f]$, where $P[f]$ is the Poisson integral (1);*
(b) *the mapping $u \mapsto L_{\tilde{\omega}}[u]$ is a linear isometry of $b_{\tilde{\omega}}^2(\mathbb{R}^n)$ onto $h^2(B)$.*

Remark 1. It is well known that for $f \in L^2(S)$ the function $P[f]$ has a nontangential limit $f(\zeta)$ at almost every point $\zeta \in S$. Thus, it is natural to identify f and $P[f]$ and to say that the operators L_ω and R_ω are mutually inverse.

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