ON THE DIVERGENCE OF TRIANGULAR AND ECCENTRICAL SPHERICAL SUMS OF DOUBLE FOURIER SERIES

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ABSTRACT. We construct a continuous function on the torus with almost everywhere divergence triangular sums of double Fourier series. An analogous theorem we also prove for eccentrical spherical sums.

1. INTRODUCTION

Carleson [3] proved that the Fourier series of any function from $L^2(\mathbb{T})$ converges almost everywhere. Hunt [6], Sjölin [13] and Antonov [1] established the same property of Fourier series in wider function classes. Now the best known result, due to Antonov [1], proves the a.e. convergence of Fourier series for the functions from $L\log L\log\log\log L(\mathbb{T})$.

The problem of almost everywhere convergence of multiple Fourier series is well investigated for different definitions of partial sums. If $f\in L^1(\mathbb{T}^2)$ is an arbitrary function with the double Fourier series

$$\sum_{n,m=-\infty}^{+\infty} c_{nm}e^{i(nx+my)} \tag{1.1}$$

and $G\subset\mathbb{R}^2$ is a bounded region, then we denote by

$$S_G(x,y,f)=\sum_{(n,m)\in G} c_{nm}e^{i(nx+my)}. \tag{1.2}$$

the partial sum of (1.1) over the region G. Let $P\subset\mathbb{R}^2$ be an arbitrary polygon containing the origin. We set

$$\lambda P=\{(\lambda x,\lambda y):\,(x,y)\in P\},\quad \lambda>0.$$

C. Fefferman [4] proved that if $f\in L^p(\mathbb{T}^2)$, $p>1$, then

$$S_{\lambda P}(x,y,f)\to f(x,y)\text{ a.e. as }\lambda\to\infty. \tag{1.3}$$

In the case when P is either rectangle or square is considered by Sjölin [13] and Antonov [1]. In the rectangle case the relation (1.3) holds for

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any $f \in L(\log L)^3 \log\log L$ ([13]). While P is a square, then it holds whenever $f \in L(\log L)^2 \log\log L$([1]). Tevzadze [14] showed that for any sequence of rectangles $R_1 \subset R_2 \subset R_3 \subset \ldots \mathbb{R}^2$ with the sides parallel to the coordinate axes the partial sums $S_{R_k}(x,y,f)$ of any function $f \in L^2(\mathbb{T}^2)$ converge a.e..

Note that in all these convergence theorems the partial sums depend on one parameter. The following theorem due to C. Fefferman [5] shows that the rectangular partial sums

$$S_{NM}(x,y,f) = \sum_{|n|\le N, |m|\le M} c_{nm} e^{i(nx+my)},$$

with two independent parameters N and M have a quite different property.

Theorem A (C. Fefferman). *There exists a real continuous function $f \in C(\mathbb{T}^2)$ such that*

$$\limsup_{N,M\to\infty} |S_{NM}(x,y,f)| = \infty$$

for any $(x,y) \in \mathbb{T}^2$.

Observe that in the above discussed convergence theorems of double Fourier series the summation regions are polygons with fixed side directions. The present paper shows that a little freedom of the side directions of the summation polygons changes the situation basically.

We consider the following rhombus regions

$$\Delta(a,b) = \left\{(x,y) \in \mathbb{R}^2 : \, a|x| + b|y| \le 1\right\}, \quad a,b > 0. \tag{1.4}$$

Given such a region $\Delta = \Delta(a,b)$, we denote

$$\rho(\Delta) = \frac{\max\{a,b\}}{\min\{a,b\}}.$$

It is clear that Δ is a square, while $\rho(\Delta) = 1$. Note that the regions

$$\Delta(a,b) \cap \mathbb{R}^2_+, \tag{1.5}$$

are triangles with a vertex at the origin. It is clear that the double series (1.1) of any real function $f \in L(\mathbb{T}^2)$ can be written in the real form (by sine and cosine functions), and the sum (1.2), corresponding to the rhombus region (1.4), coincides with the partial sum of the Fourier series in the real form over the triangle (1.5).

A sequence of regions G_k is said to be complete, if $\cup_{k=1}^\infty G_k = \mathbb{R}^2$.

The next theorem is an equivalent reformulation of the theorem of C. Fefferman [4].

Theorem B (C. Fefferman). *If Δ_k, $k = 1, 2, \ldots$, is a complete increasing sequence of squares of the form (1.4 ($\rho(\Delta_k) = 1$), then for any function $f \in L^p(\mathbb{T}^2)$, $p > 1$, the relation*

$$\lim_{k\to\infty} S_{\Delta_k}(x, y, f) = f(x, y)$$

holds almost everywhere.

In the present paper we prove the following theorem, which shows that in Theorem B the condition $\rho(\Delta_k) = 1$ can not be replaced by $\rho(\Delta_k) \to 1$.

Theorem 1. *There exists a real continuous function $f \in C(\mathbb{T}^2)$ and a complete sequence of regions Δ_k, $k = 1, 2, \ldots,$ of the form (1.4 such that $\Delta_k \subset \Delta_{k+1}$, $\rho(\Delta_k) \to 1$ and*

$$\limsup_{k\to\infty} |S_{\Delta_k}(x, y, f)| = \infty \tag{1.6}$$

almost everywhere.

An example of a function $f \in L^p(\mathbb{T}^2)$, $1 \le p < \infty$, satisfying the same relation (1.6) was constructed in [8]. An analogous divergence theorem for Walsh-Fourier series was considered in [9].

We obtain also a similar divergence theorem for some spherical sums. Let $B = B(x_0, y_0, r)$ be the open ball, with the radius r and the center at the point (x_0, y_0). We define the following quantity

$$\tau(B) = \frac{\sqrt{x_0^2 + y_0^2}}{r},$$

describing the eccentricity of the ball against the origin. We prove

Theorem 2. *There exists a continuous function $f \in C(\mathbb{T}^2)$ and a complete sequence of balls U_k, $k = 1, 2, \ldots,$ such that $\tau(U_k) \to 0$ and*

$$\limsup_{k\to\infty} |S_{U_k}(x, y, f)| = \infty \tag{1.7}$$

almost everywhere.

In the proofs of the theorems we use the method applied in the paper [7], where we establish the unboundedness of the maximal directional Hilbert transform on the plane, associated with an arbitrary infinite family of directions.

Unfortunately, we are not able to prove Theorem 2 with the condition $\tau(U_k) = 0$ instead of $\tau(U_k) \to 0$. That would be a negative answer to the well known problem on almost everywhere convergence of spherical partial sums of double Fourier series.

2. Auxiliary lemmas

Let $\mathbb{T}=\mathbb{R}/(2\pi\mathbb{Z})$ be the one dimensional torus and $\mathbb{T}^2=\mathbb{T}\times\mathbb{T}$. If E is a Lebesgue measurable set in $\mathbb{T}$ or $\mathbb{T}^2$, then the notation $\mathbb{I}_E$ stands for the indicator function of E, $|E|$ denotes the Lebesgue measure of E. For any $n\in\mathbb{N}$ and for a measurable set $E\subset\mathbb{T}^2$ we set

$$E(n)=\{(x,y)\in\mathbb{T}^2:\,(nx,ny)\in E\}.$$

It is clear that $|E(n)|=|E|$. The relation

$$\lim_{n\to\infty}|A\cap B(n)|=\frac{|A||B|}{4\pi^2},\quad (4\pi^2=|\mathbb{T}^2|), \tag{2.1}$$

is well known and follows from a theorem of Fejèr (see for example [15], ch. 2, Theorem (4.15)). The following two lemmas are based on a standard probabilistic independence argument.

Lemma 1. *Let $n_0>0$ be an arbitrary integer and $0<\alpha<1$. Then for any sequence of measurable sets $E_k\subset\mathbb{T}^2$, $k=1,2,\ldots,l$, with $|E_k|>4\pi^2\alpha$, there exist natural numbers $n_0<n_1<n_2<\ldots<n_l$ satisfying the condition*

$$\left|\bigcup_{k=1}^{l}E_k(n_k)\right|>4\pi^2(1-(1-\alpha)^l). \tag{2.2}$$

Proof. From (2.1) it follows that

$$\begin{aligned}\lim_{n\to\infty}|(A\cup B(n))^c|&=\lim_{n\to\infty}\left(4\pi^2-|A|-|B(n)|+|A\cap B(n)|\right)\\&=4\pi^2-|A|-|B|+\frac{|A||B|}{4\pi^2}=\frac{(4\pi^2-|A|)(4\pi^2-|B|)}{4\pi^2}\\&=\frac{|A^c|\cdot|B^c|}{4\pi^2}.\end{aligned}$$

Taking small enough $\delta>0$, then applying this relation successively $l-1$ time, we may find integers $1=n_1<n_2<\ldots<n_l$ such that

$$\begin{aligned}|(E_1\cup E_2(n_2)\cup\ldots\cup E_l(n_l))^c|&<\frac{|E_1^c|\cdot|E_2^c|\cdot\ldots\cdot|E_l^c|}{(4\pi^2)^{l-1}}+\delta\\&<\frac{(4\pi^2-4\pi^2\alpha)^l}{(4\pi^2)^{l-1}}=4\pi^2(1-\alpha)^l.\end{aligned}$$

This immediately gives (2.2). □

Lemma 2. *Let $E_k\subset\mathbb{T}^2$ be a sequence of measurable sets such that $|E_k|>4\pi^2\alpha$, $k=1,2,\ldots$, where $0<\alpha<1$. Then there exists an* *infinite sequence of integers* $0 < n_1 < n_2 < \dots$ *such that*

$$\left|\bigcap_{l\geq 1}\bigcup_{k\geq l} E_k(n_k)\right| = 4\pi^2 = |\mathbb{T}^2|. \tag{2.3}$$

Proof. Applying Lemma 1 to the set families

$$\left\{E_j : k^2 < j \leq (k+1)^2\right\}, \quad k = 0, 1, 2, \dots,$$

we find integers n_j satisfying

$$\left|\bigcup_{j=k^2+1}^{(k+1)^2} E_j(n_j)\right| > 4\pi^2(1 - (1-\alpha)^{2k+1}), \quad k = 0, 1, \dots .$$

Thus we get $|\cup_{k\geq l} E_k(n_k)| = 4\pi^2$ for any $l = 1, 2, \dots$, and so (2.3). □

Lemma 3. *For any function* $f \in L^\infty(\mathbb{T}^2)$ *we have*

$$|\{(x,y) \in \mathbb{T}^2 : |f(x,y)| > \|f\|_1/8\pi^2\}| \geq \frac{4\pi^2}{8\pi^2(\|f\|_\infty/\|f\|_1) - 1}. \tag{2.4}$$

Proof. Denote

$$E = \{(x,y) \in \mathbb{T}^2 : |f(x,y)| > \|f\|_1/8\pi^2\}.$$

We have

$$\|f\|_1 \leq (4\pi^2 - |E|)\frac{\|f\|_1}{8\pi^2} + |E|\|f\|_\infty.$$

After a simple transformation from this inequality we get (2.4). □

For any integer $n > 2$ of the form

$$n = 2^k + j, \quad 1 \leq j \leq 2^k, \quad k = 1, 2, \dots,$$

we denote

$$\bar{n} = 2^{k-1} + \left[\frac{j+1}{2}\right], \tag{2.5}$$

where $[\cdot]$ stands for the integer part of a number. A sequence of real valued functions $f_n(x,y)$, $n = 2, 3, \cdots, 2^m$, ($f_n \not\equiv 0$) is said to be a tree-system if

$$\operatorname{supp} f_n \subset \{(x,y) \in \mathbb{T}^2 : (-1)^{j+1} \cdot f_{\bar{n}}(x) > 0\}.$$

The Haar system excluded the first function is the typical example of a tree-system. The following lemma was proved in [7]. Its Haar system case was considered in [11].

Lemma 4. *There exists a rearrangement σ of the integers $\{2, 3, \cdots, 2^m\}$ such that for any tree system $f_n(x, y)$, $n = 2, 3, \cdots, 2^m$, we have*

$$\sup_{2\le l\le 2^m}\left|\sum_{n=2}^{l} f_{\sigma(n)}(x,y)\right| \ge \frac{1}{3}\sum_{n=2}^{2^m}\left|f_n(x,y)\right|.$$

For any p integer we denote

$$\delta_p^k = \left(\frac{2\pi(k-1)}{|p|}, \frac{2\pi k}{|p|}\right),$$

$$\delta_p^{k,j} = \delta_p^k \times \delta_p^j = \left(\frac{2\pi(k-1)}{|p|}, \frac{2\pi k}{|p|}\right) \times \left(\frac{2\pi(j-1)}{|p|}, \frac{2\pi j}{|p|}\right),\ k, j \in \mathbb{Z},$$

$$Q_p = \{\delta_p^{k,j} : 1 \le k, j \le |p|\}.$$

Given number $\gamma > 1$ and sequence of integers p_n, $n = 2, 3, \dots, \nu = 2^m$, such that $1 \le |p_2| < |p_3| < \dots < |p_\nu|$ and $p_n|p_{n+1}$(p_n divides p_{n+1}) we associate the function system

$$a_n(x,y) = a_n(x,y) = \frac{\mathbb{I}_{E_n}(x,y)}{\sqrt{m}} e^{i(p_n x + q_n y)}, \quad n \ge 2, \tag{2.6}$$

where $E_2 = \mathbb{T}^2$ and E_n is defined to be the union of all rectangles $\delta \in Q_{p_n}$ satisfying the conditions

$$\overline{\delta} \subset \{(x,y) \in E_{\bar n} : (-1)^{j+1}\cos(p_{\bar n}x + q_{\bar n}y) > 0\}, \tag{2.7}$$

$$\sup_{(x,y)\in\delta}\left|\sum_{k=2}^{\bar n} a_k(x,y) + \frac{e^{i(p_n x + q_n y)}}{\sqrt{m}}\right| \le \gamma. \tag{2.8}$$

Note that some of the sets E_n can be empty. Besides, for the further convenience, we also assume that

$$E_n = \varnothing, \quad n > \nu = 2^m.$$

For any $k = 1, 2, \dots, m-1$ we consider the collection

$$\mathcal{E}_k = \{E_n : 2^k < n \le 2^{k+1}\}.$$

The following relations give structural characterization of the sets E_n:

$$E_n \cap E_{n'} = \varnothing, \quad 2^k < n < n' \le 2^{k+1}, \tag{2.9}$$

$$E_n \in \mathcal{E}_k \Rightarrow E_{2n-1}, E_{2n} \in \mathcal{E}_{k+1}, \tag{2.10}$$

$$E_{2n-1} \cup E_{2n} \subset E_n. \tag{2.11}$$

Obviously the relations (2.6)-(2.8) define the system (2.6) uniquely. From (2.9) and (2.11) we obtain

$$\sum_{n=2}^{\nu} |a_n(x,y)| \le \sum_{n=2}^{\nu} \frac{\mathbb{I}_{E_n}(x,y)}{\sqrt{m}} \le \sqrt{m}. \tag{2.12}$$

Thus we conclude that if $\gamma \geq \sqrt{m}$, then the condition (2.8) holds for any δ, and so $a_n(x,y)$ doesn't depend on γ. In this case the set E_n and the function $a_n(x,y)$ will be denoted by F_n and $b_n(x,y)$ respectively.

Lemma 5. *If $p_n|p_{n+1}$ and $|p_{n+1}| \geq |p_n|\sqrt{m}$, $n = 2,3,\ldots,\nu-1$, then*

$$\left|\left\{\max_{2\leq n\leq\nu}\left|\sum_{k=2}^{n} b_k(x,y)\right| > \lambda\right\}\right| \leq \frac{c}{\lambda}, \quad \lambda > 0, \tag{2.13}$$

where $c > 0$ is an absolute constant.

Proof. Without loss of generality we can assume that $p_n > 0$, $n = 2,3,\ldots,\nu$. Since F_n consists of squares from Q_{p_n}, from (2.6) we get

$$\int_{\delta^j_{p_n}} b_n(t,y)dt = 0.$$

Define

$$f_n(x,y) = \frac{p_{n+1}}{2\pi}\sum_{j=1}^{p_{n+1}}\left[\int_{\delta^j_{p_{n+1}}} b_n(t,y)dt \cdot \mathbb{I}_{\delta^j_{p_{n+1}}}(x)\right] \tag{2.14}$$

for $n = 2,3,\ldots,\nu$, where $p_{\nu+1} > p_\nu\sqrt{m}$ is taken arbitrarily such that $p_\nu|p_{\nu+1}$. From the definition of $b_n(x,y)$ (see (2.6)-(2.8) in the case $\gamma > \sqrt{m}$) we conclude

$$\operatorname{supp} f_n \subset \operatorname{supp} b_n \subset F_n, \quad n = 2,3,\ldots,\nu. \tag{2.15}$$

For a fixed y the function $f_n(x,y)$ is constant on each interval $\delta^j_{p_{n+1}}$, $j = 1,2,\ldots,p_{n+1}$, with respect to the variable x and we have

$$\begin{aligned}\int_{\delta^j_{p_{n+1}}} f_{n+1}(t,y)dt &= \sum_{k:\,\delta^k_{p_{n+2}}\subset\delta^j_{p_{n+1}}}\int_{\delta^k_{p_{n+2}}} b_{n+1}(t,y)dt \\ &= \int_{\delta^j_{p_{n+1}}} b_{n+1}(t,y)dt = 0.\end{aligned}$$

This means that the functions (2.14) form a martingale difference sequence with respect to x. Thus, applying a well known martingale inequality (see [10], ch. IV.5.1) and then (2.9) and (2.11), we obtain

$$\begin{aligned}\int_{\mathbb{T}^2}\max_{2\leq n\leq\nu}\left|\sum_{k=2}^{n} f_k(x,y)\right|^2 dxdy &\lesssim \sum_{k=2}^{\nu}\|f_k\|_2^2 \leq \sum_{k=2}^{\nu}\|b_k\|_2^2 \\ &\leq \frac{1}{m}\sum_{k=2}^{\nu}|E_k| \leq 1.\end{aligned} \tag{2.16}$$

On the other hand if $x\in \delta^j_{p_{k+1}}$, then

$$|f_k(x,y)-b_k(x,y)|\le \frac{p_{k+1}}{2\pi}\int_{\delta^j_{p_{k+1}}}|b_k(t,y)-b_k(x,y)|dt \tag{2.17}$$

$$\le \sup_{|t-x|\le 2\pi/p_{k+1}}\frac{|\cos p_kt-\cos p_kx|}{\sqrt{m}}\le \frac{2\pi p_k}{p_{k+1}\sqrt{m}}\le\frac{2\pi}{m}. \tag{2.18}$$

Combining this with (2.12) and (2.15), we get

$$\sum_{k=2}^{\nu}|f_k(x,y)-b_k(x,y)|\le\frac{2\pi}{m}\sum_{k=2}^{\nu}\mathbb{I}_{F_k}(x,y)\le 2\pi.$$

This together with (2.16) derives

$$\int_{\mathbb{T}^2}\max_{2\le n\le\nu}\left|\sum_{k=2}^{n}b_k(x,y)\right|^2dxdy$$
$$\le 2\int_{\mathbb{T}^2}\max_{2\le n\le\nu}\left|\sum_{k=2}^{n}f_k(x,y)\right|^2dxdy+8\pi^2\lesssim 1.$$

Then, using Chebyshev's inequality, from this we will get (2.13). □

Lemma 6. *Let c be the constant from (2.13) and $\gamma\ge c+2$. Then if $2p_{\bar n}|p_n$ and*

$$|p_n|>20\nu(|p_{\bar n}|+|q_{\bar n}|), \tag{2.19}$$

then

$$\sum_{n=2}^{\nu}\int_{\mathbb{T}^2}|\mathrm{Re}\,(a_n(x,y))|dxdy>2\sqrt{m}. \tag{2.20}$$

Proof. Observe that we may assume $p_n,q_n>0$, $n=2,3,\ldots,\nu$. The proof of (2.20) is based on the bound

$$\left|\bigcup_{n=2^{m-1}+1}^{2^m}E_n\right|>10. \tag{2.21}$$

In order to prove (2.21) we define the sets U_n and V_n to be the union of all squares $\delta\in Q_{p_n}$ satisfying the condition

$$\bar\delta\subset\{(x,y)\in E_{\bar n}:\,(-1)^{j+1}\cos(p_{\bar n}x+q_{\bar n}y)>0\}, \tag{2.22}$$

$$\bar\delta\cap\{(x,y)\in E_{\bar n}:\,(-1)^{j+1}\cos(p_{\bar n}x+q_{\bar n}y)>0\}\ne\varnothing, \tag{2.23}$$

respectively. We claim the following relations:

$$E_n \subset U_n \subset E_{\bar{n}} \cap V_n, \tag{2.24}$$

$$E_n \subset \overline{V_{2n}} \cup \overline{V_{2n-1}}, \tag{2.25}$$

$$|V_n \setminus U_n| < \frac{1}{\nu}, \tag{2.26}$$

$$U_n \setminus E_n \subset \left\{ (x,y) \in \mathbb{T}^2 : \max_{1\le n\le \nu} \left| \sum_{k=1}^{n} a_k(x,y) \right| > \gamma - 2 \right\}. \tag{2.27}$$

The inclusion (2.24) immediately follows from the definitions of E_n, U_n and V_n (see (2.7), (2.22), (2.23)).

From (2.5) it easily follows that $\overline{2n-1} = \overline{2n} = n$. Thus by the definition V_{2n} and V_{2n-1} are the union of squares $\delta \in Q_{p_n}$ satisfying respectively

$$\bar{\delta} \cap \{(x,y) \in E_n : \cos(p_{\bar{n}}x + q_{\bar{n}}y) > 0\} \neq \varnothing,$$
$$\bar{\delta} \cap \{(x,y) \in E_n : \cos(p_{\bar{n}}x + q_{\bar{n}}y) < 0\} \neq \varnothing.$$

This implies (2.25).

To prove (2.26) we note that the set $V_n \setminus U_n$ $(n = 2^k + j)$ consists of the squares $\delta \in Q_{p_n}$ satisfying

$$\bar{\delta} \cap \{(x,y) \in E_{\bar{n}} : (-1)^{j+1}\cos(p_{\bar{n}}x + q_{\bar{n}}y) = 0\} \neq \varnothing. \tag{2.28}$$

Then observe that if

$$\delta_{p_n}^{\alpha_1,\beta} \subset V_n \setminus U_n \tag{2.29}$$

and the integer α_2 satisfies

$$\alpha_1 + \frac{p_n}{2p_{\bar{n}}} \cdot \frac{1}{3\nu} + 1 < \alpha_2 < \alpha_1 + \frac{p_n}{2p_{\bar{n}}} \left(1 - \frac{1}{3\nu}\right) - 1, \tag{2.30}$$

then

$$\delta_{p_n}^{\alpha_2,\beta} \cap (V_n \setminus U_n) = \varnothing. \tag{2.31}$$

Indeed, suppose we have (2.29), (2.30), and besides (2.31) doesn't hold. Then according to (2.28) there are points

$$(x_1, y_1) \in \overline{\delta_{p_n}^{\alpha_1,\beta}}, \quad (x_2, y_2) \in \overline{\delta_{p_n}^{\alpha_2,\beta}}, \tag{2.32}$$

such that $\cos(p_{\bar{n}}x_1 + q_{\bar{n}}y_1) = \cos(p_{\bar{n}}x_2 + q_{\bar{n}}y_2) = 0$ and therefore

$$p_{\bar{n}}x_1 + q_{\bar{n}}y_1 = \frac{\pi}{2} + \pi l_1, \quad p_{\bar{n}}x_2 + q_{\bar{n}}y_2 = \frac{\pi}{2} + \pi l_2,$$

for some integers l_1 and l_2. Thus we will get

$$p_{\bar{n}}(x_2 - x_1) + q_{\bar{n}}(y_2 - y_1) = \pi(l_2 - l_1). \tag{2.33}$$

From (2.19), (2.30) and (2.32) we derive

$$|p_{\bar n}(x_2-x_1)+q_{\bar n}(y_2-y_1)|\le \frac{2\pi p_{\bar n}(\alpha_2-\alpha_1+1)}{p_n}+\frac{2\pi q_{\bar n}}{p_n} < \pi\left(1-\frac{1}{3\nu}\right)+\frac{\pi}{10\nu}<\pi. \tag{2.34}$$

Combining (2.33) and (2.34), we get $l_1=l_2$ and therefore

$$p_{\bar n}(x_2-x_1)=q_{\bar n}(y_1-y_2). \tag{2.35}$$

On the other hand, using (2.19), (2.30) and (2.32), we have

$$q_{\bar n}(y_1-y_2)\le \frac{2\pi q_{\bar n}}{p_n}<\frac{\pi}{10\nu} \tag{2.36}$$

and

$$p_{\bar n}(x_2-x_1)>\frac{2\pi p_{\bar n}(\alpha_2-\alpha_1-1)}{p_n}>\frac{2\pi p_{\bar n}}{p_n}\frac{p_n}{2p_{\bar n}}\cdot\frac{1}{3\nu}=\frac{\pi}{3\nu}. \tag{2.37}$$

Combining (2.35)-(2.37) we get contradiction. Hence we have

$$(2.29, (2.30 \Rightarrow (2.31. \tag{2.38}$$

For $1\le k\le p_n$ and $1\le k\le p_{\bar n}$ we consider the squares

$$\left\{\delta_{p_n}^{k,j}:\ \frac{(k-1)p_n}{2p_{\bar n}}<j\le\frac{kp_n}{2p_{\bar n}}\right\}. \tag{2.39}$$

Since $2p_{\bar n}$ divides p_n, either they are all inside of $E_{\bar n}$ or all are outside of $E_{\bar n}$. Using the relations (2.19) and (2.38), one can easily conclude that the number of squares $\delta_{p_n}^{k,j}$ from the collection (2.39) included in $V_n\setminus U_n$ doesn't exceed the quantity

$$\frac{2}{3\nu}\cdot\frac{p_n}{2p_{\bar n}}+3<\frac{1}{\nu}\cdot\frac{p_n}{2p_{\bar n}}.$$

Thus we get the number of all squares $\delta\in Q_{p_n}$ with $\delta\subset V_n\setminus U_n$ is estimated above by the value $(p_n)^2/\nu$, where $(p_n)^2$ is the number of all squares $\delta\in Q_{p_n}$. Hence we get

$$|V_n\setminus U_n|<\frac{(p_n)^2}{\nu}\cdot\frac{4\pi^2}{(p_n)^2}=\frac{1}{\nu}.$$

If (x,y) belong to the left side of (2.27), then according to the definitions of U_n ((2.22)) and E_n ((2.7), (2.8))), there exists a unique $\delta\in Q_{p_n}$ such that $(x,y)\in\delta\subset U_n$ and $\delta\cap E_n=\varnothing$. From (2.22) we have $\delta\subset E_{\bar n}$, then using (2.7) and (2.8), we conclude

$$\sup_{(u,v)\in\delta}\left|\sum_{k=1}^{\bar n}a_k(u,v)+\frac{e^{i(p_nu+q_nv)}}{\sqrt{m}}\right|>\gamma. \tag{2.40}$$

By (2.40) there exists a point $(x_0, y_0) \in \delta$ satisfying

$$\left|\sum_{k=1}^{\bar{n}} a_k(x_0, y_0) + \frac{e^{i(p_n x_0 + q_n y_0)}}{\sqrt{m}}\right| > \gamma. \tag{2.41}$$

On the other hand for an arbitrary $(x, y), (x', y') \in \delta$ we have

$$\begin{aligned}
&\left|e^{i(p_{\bar{n}}x + q_{\bar{n}}y)} - e^{i(p_{\bar{n}}x' + q_{\bar{n}}y')}\right| \\
&\qquad \leq \left|e^{i(p_{\bar{n}}x + q_{i\bar{n}}y)} - e^{i(p_{\bar{n}}x + q_{i\bar{n}}y')}\right| + \left|e^{i(p_{\bar{n}}x + q_{\bar{n}}y')} - e^{i(p_{\bar{n}}x' + q_{\bar{n}}y')}\right| \\
&\qquad = \left|e^{iq_{\bar{n}}y} - e^{iq_{\bar{n}}y'}\right| + \left|e^{ip_{\bar{n}}x} - e^{ip_{\bar{n}}x'}\right| \\
&\qquad \leq \sqrt{2}(q_{\bar{n}}|y - y'| + p_{\bar{n}}|x - x'|) \\
&\qquad \leq \frac{4\pi(p_{\bar{n}} + q_{\bar{n}})}{p_n} < \frac{1}{\nu}
\end{aligned}$$

and therefore

$$r = \sup_{(x,y),(x',y') \in \delta} \left|\sum_{k=1}^{\bar{n}} a_k(x, y) - \sum_{k=1}^{\bar{n}} a_k(x', y')\right| \leq 1.$$

Thus, using (2.41), we obtain

$$\begin{aligned}
\left|\sum_{k=1}^{\bar{n}} a_k(x, y)\right| &\geq \left|\sum_{k=1}^{\bar{n}} a_k(x_0, y_0)\right| - r \\
&\geq \left|\sum_{k=1}^{\bar{n}} a_k(x_0, y_0) + \frac{e^{p_n x_0 + q_n y_0}}{\sqrt{m}}\right| - \frac{|e^{p_n x_0 + q_n y_0}|}{\sqrt{m}} - 1 \\
&> \gamma - 2,
\end{aligned}$$

which gives (2.27). According to (2.25) we have

$$E_n \setminus (E_{2n-1} \cup E_{2n}) \subset (\overline{V_{2n-1}} \setminus E_{2n-1}) \cup (\overline{V_{2n}} \setminus E_{2n})$$

Using this we get

$$\begin{aligned}
\sum_{n=2^{m-1}+1}^{2^m} E_n &= \mathbb{T}^2 \setminus \bigcup_{n=2}^{2^{m-1}} (E_n \setminus (E_{2n-1} \cup E_{2n})) \\
&\supset \mathbb{T}^2 \setminus \bigcup_{n=3}^{\nu} (\overline{V_n} \setminus E_n) \\
&= \mathbb{T}^2 \setminus \left(\bigcup_{n=3}^{\nu} (\overline{V_n} \setminus U_n) \cup (U_n \setminus E_n)\right).
\end{aligned} \tag{2.42}$$

From (2.26) we get

$$\left|\bigcup_{n=3}^{\nu}\overline{V_n}\setminus U_n\right| = \left|\bigcup_{n=3}^{\nu}V_n\setminus U_n\right| \le 1.$$

Applying Lemma 6, from (2.27) it follows that

$$\begin{aligned}\left|\bigcup_{n=3}^{\nu}(U_n\setminus E_n)\right| \\ \le \left|\left\{(x,y)\in\mathbb{T}^2:\ \max_{1\le n\le\nu}\left|\sum_{k=1}^{n}a_k(x,y)\right|>\gamma-2\right\}\right| \\ \le \frac{c}{\gamma-2}\le\frac{c+2}{\gamma}<1.\end{aligned} \tag{2.43}$$

Combing (2.42)-(2.43) we obtain

$$\left|\sum_{n=2^{m-1}+1}^{2^m}E_n\right| \ge |\mathbb{T}^2|-2\ge 10$$

and so (2.21). Using the properties of the sets E_n ((2.9)-(2.11)), from (2.21) we conclude

$$\sum_{n=2^{k-1}+1}^{2^k}|E_n| \ge \sum_{n=2^{m-1}+1}^{2^m}|E_n| > 10$$

for any $k=1,2,\dots,m$ and therefore

$$\sum_{n=2}^{\nu}|E_n|\ge 10m. \tag{2.44}$$

On the other hand, since E_n is a union of squares $\delta \in Q_{p_n}$, we have

$$\begin{aligned}
(2.45)\quad \int_{E_n} |\cos(p_n x + q_n y)| dxdy & \\
&= \sum_{\delta\in Q_{p_n},\, \delta\subset E_n} \int_\delta |\cos(p_n x + q_n y)| dxdy \\
&= \sum_{\delta\in Q_{p_n},\, \delta\subset E_n} \frac{1}{p_n q_n} \int_{\mathbb{T}^2} |\cos(x + y)| dxdy \\
&= 4\pi \sum_{\delta\in Q_{p_n},\, \delta\subset E_n} \frac{1}{p_n q_n} \\
&= \frac{1}{\pi} \sum_{\delta\in Q_{p_n},\, \delta\subset E_n} |\delta| = \frac{|E_n|}{\pi}.
\end{aligned}$$

Combining (2.44) and (2.45) we obtain

$$\begin{aligned}
\sum_{n=2}^{\nu} \int_{\mathbb{T}^2} |\mathrm{Re}\,(a_n(x,y))| dxdy &= \frac{1}{\sqrt{m}} \sum_{n=2}^{\nu} \int_{E_n} |\cos(p_n x + q_n y)| dxdy \\
&\geq \frac{1}{\pi\sqrt{m}} \sum_{n=2}^{\nu} |E_n| \geq 2\sqrt{m}.
\end{aligned}$$

□

Lemma 7. *Let σ be the rearrangement of the integers $\{2, 3, \cdots, 2^m\}$ determined by Lemma 4. If γ and p_n satisfy the hypothesis of Lemma 6, then*

$$(2.46)\qquad \left| \left\{ (x,y) \in \mathbb{T}^2 : \max_{2\leq n\leq \nu} \left| \sum_{j=1}^{n} a_{\sigma(j)}(x,y) \right| > \frac{\sqrt{m}}{120} \right\} \right| > 1.$$

Proof. The function system

$$u_n(x,y) = \frac{\mathbb{I}_{E_n}(x,y)}{\sqrt{m}} \cos(p_n x + q_n y) = \mathrm{Re}\,(a_n(x,y)),$$

where $n = 2, 3, \cdots, \nu = 2^m$, is a tree-system, since by definition ((2.7), (2.8)) we have

$$\begin{aligned}
\mathrm{supp}\,(u_n(x,y)) \subset E_n \subset \big\{ (x,y) \in E_{\bar{n}} :\, (-1)^{j+1} \cos(p_{\bar{n}} x + q_{\bar{n}} y) > 0 \big\} \\
= \big\{ (x,y) \in E_{\bar{n}} :\, (-1)^{j+1} u_{\bar{n}}(x,y) > 0 \big\} \\
= \big\{ (x,y) \in \mathbb{T}^2 :\, (-1)^{j+1} u_{\bar{n}}(x,y) > 0 \big\},
\end{aligned}$$

Hence, applying Lemma 4, we get

$$\sup_{2\le l\le\nu}\left|\sum_{n=2}^{l} a_{\sigma(n)}(x,y)\right| \tag{2.47}$$
$$\ge \sup_{2\le l\le\nu}\left|\sum_{n=2}^{l} u_{\sigma(n)}(x,y)\right| \ge \frac{1}{3}\sum_{n=2}^{\nu}\left|u_n(x,y)\right|.$$

Consider the function

$$f(x,y)=\sum_{n=2}^{\nu}|u_n(x,y)|. \tag{2.48}$$

By Lemma 6 we have $\|f\|_1 > 2\sqrt{m}$. On the other hand $\|f\|_\infty \le \sqrt{m}$, because

$$0\le f(x,y)\le \sum_{n=2}^{\nu}\frac{\mathbb{I}_{E_n}(x,y)}{\sqrt{m}}\le\sqrt{m}.$$

Thus, applying Lemma 3, we get

$$\left|\left\{(x,y)\in\mathbb{T}^2:\,|f(x,y)|>\frac{\sqrt{m}}{4\pi^2}\right\}\right| > \frac{4\pi^2}{8\pi^2(\|f\|_\infty/\|f\|_1)-1}$$
$$\ge\frac{4\pi^2}{4\pi^2-1}>1.$$

Combining this with (2.47) and (2.48), we obtain (2.46). □

Lemma 8. *If $\gamma>1$ and p_n divides p_{n+1}, then*

$$\left|\sum_{k=2}^{\nu} a_k(x,y)\right|\le\gamma,\quad (x,y)\in\mathbb{T}^2.$$

Proof. Take an arbitrary $(x,y)\in\mathbb{T}^2$. We have

$$(x,y)\in E_n\setminus(E_{2n-1}\cup E_{2n}) \tag{2.49}$$

for an integer $2\le n\le 2^m$. Then from the definition of the sets E_n ((2.7), (2.8)) we get

$$\left|\sum_{k=2}^{n} a_k(x,y)\right|\le\gamma,$$

and $a_k(x,y)=0$ while $k>n$. This implies

$$\left|\sum_{k=2}^{\nu} a_k(x,y)\right|=\left|\sum_{k=2}^{n} a_k(x,y)\right|\le\gamma,$$

and so the lemma is proved. □

Any finite sum of the form

$$T(x,y)=\sum_{(n,m)\in G} c_{nm}e^{i(nx+my)},$$

where $G\subset\mathbb{R}^2$ is a bounded region, is said to be a double trigonometric polynomial. The spectrum of this polynomial is denoted by

$$\mathrm{spec}\,(T)=\left\{(n,m)\in\mathbb{Z}^2:\ c_{nm}\neq 0\right\}.$$

We will consider the sectorial regions

$$V(\alpha,\beta)=\{(x,y)\in\mathbb{R}^2:\ x=r\cos\theta, y=r\sin\theta,\\ r\ge 0,\ \alpha\le\theta<\beta\}, \tag{2.50}$$

where $0\le\alpha<\beta\le 2\pi$. In the proof of the following basic lemma the technique of the paper [7] is used.

Lemma 9. *If S_n, $n=2,3,\cdots,\nu=2^m$, $m\ge 10$, is an arbitrary sequence of sectors of the form (2.50), then there exists a sequence of polynomials $T_n(x,y)$, $n=2,3,\cdots,\nu$, such that*

$$\mathrm{spec}\,T_n\subset S_n,\quad n=2,3,\cdots,\nu, \tag{2.51}$$

$$\left\|\sum_{n=1}^{\nu}T_n\right\|_\infty\le c_1, \tag{2.52}$$

$$\left|\left\{(x,y)\in\mathbb{T}^2:\ \max_{2\le n\le\nu}\left|\sum_{j=1}^{n}T_j(x,y)\right|>c_2\sqrt{m}\right\}\right|>1, \tag{2.53}$$

where $c_1,c_2>0$ are some absolute constants.

Proof. Let σ be the rearrangement of the numbers $\{2,3,\cdots,\nu=2^m\}$ determined by Lemma 4 and let

$$\gamma=c+2,$$

where c is the constant from (2.13). We define positive integers p_n,q_n and double trigonometric polynomials $f_n(x,y)$ such that they, together with the sets $E_n\subset\mathbb{T}^2$, $n=2,3,\cdots\nu$, defined by (2.7) and (2.8) satisfy the relations

(2.54) $2p_{\bar{n}}|p_n$,

(2.55) $|p_n| > 20m(|p_{\bar{n}}| + |q_{\bar{n}}|)$,

(2.56) $\text{spec}\ \big(f_n(x,y)e^{i(p_nx+q_ny)}\big) \subset S_{\sigma^{-1}(n)}$, $2 \le n \le \nu$,

(2.57) $0 \le f_n(x,y) \le \dfrac{1}{\sqrt{m}}, \quad (x,y) \in \mathbb{T}^2$,

(2.58) $\left| f_n(x,y) - \dfrac{\mathbb{I}_{E_n}(x,y)}{\sqrt{m}} \right| < \dfrac{1}{\nu}, \quad (x,y) \in E_n \cup (E_{\bar{n}})^c,\ 2 \le n \le \nu.$

We will use the induction. As a first step of induction we suppose $E_2 = \mathbb{T}^2$, $f_2(x,y) \equiv 1/\sqrt{m}$ and fix integers p_2 and q_2 with $(p_2, q_2) \in S_{\sigma^{-1}(2)}$. Obviously we will have the relations (2.54)-(2.58) for $n = 2$. Now we suppose that the conditions (2.54)-(2.58) are satisfied for any $n < l$ and in particular for $\bar{l}$. Since $\overline{E}_l \subset E_{\bar{l}}$ and $E_{\bar{l}}$ is an open set, we may find a polynomial $f_l(x,y)$ satisfying (2.57) and (2.58) (with $n = l$). Since

$$\text{spec}\ \big(f_n(x,y)e^{i(p_nx+q_ny)}\big) = \text{spec}\ (f_n) + (p_n, q_n),$$

we can choose integers p_n and q_n satisfying (2.54)-(2.56) $(n = l)$. This completes the induction. Now we define our desired polynomials as follows:

$$T_n(x,y) = f_{\sigma(n)}(x,y)e^{i(p_{\sigma(n)}x+q_{\sigma(n)}y)},\ 2 \le n \le \nu.$$

Together with $T_n(x,y)$ we will consider also the function system $a_n(x,y)$ defined in (2.6). From (2.56) we have

$$\text{spec}\,(T_n) = \text{spec}\ \big(f_{\sigma(n)}(x,y)e^{i(p_{\sigma(n)}x+q_{\sigma(n)}y)}\big) \subset S_n,$$

which implies (2.51). If $(x,y) \in \mathbb{T}^2$, then we have

$$(x,y) \in E_n \setminus (E_{2n-1} \cup E_{2n})$$

for some integer $2 \le n \le 2^m$. From this it follows that $(x,y) \in E_j \cup (E_{\bar{j}})^c$ whenever $2 \le j \le \nu$ and $j \ne 2n, 2n-1$. Thus we get

$$\left| f_j(x,y) - \frac{\mathbb{I}_{E_j}(x,y)}{\sqrt{m}} \right| < \frac{1}{\nu}, \quad 2 \le j \le \nu, \quad j \ne 2n, 2n-1,$$

and therefore

$$\begin{aligned}(2.59)\qquad \sum_{j=1}^{\nu}\left|T_j(x,y)-a_{\sigma(n)}(x,y)\right| \\ &= \sum_{j=1}^{\nu}\left|f_j(x,y)-\frac{\mathbb{I}_{E_j}(x,y)}{\sqrt{m}}\right| \\ &\le \sum_{j\neq 2n,2n-1}\left|f_j(x,y)-\frac{\mathbb{I}_{E_j}(x,y)}{\sqrt{m}}\right|+1 \\ &\le 2.\end{aligned}$$

From this and Lemma 8 we get

$$\left|\sum_{n=1}^{\nu}T_n(x,y)\right| \le \left|\sum_{n=1}^{\nu}a_n(x,y)\right|+2\le \gamma+2=c+4$$

and hence (2.52). From (2.59) it also follows that

$$\begin{aligned}(2.60)\qquad \max_{2\le l\le\nu}\left|\sum_{j=2}^{l}T_j(x,y)\right| &\ge \max_{2\le l\le\nu}\left|\sum_{j=2}^{l}a_{\sigma(j)}(x,y)\right| \\ &\quad -\sum_{j=2}^{\nu}\left|T_j(x,y)-a_{\sigma(j)}(x,y)\right| \\ &\ge \max_{2\le l\le\nu}\left|\sum_{j=2}^{l}a_{\sigma(j)}(x,y)\right|-2.\end{aligned}$$

Combining (2.60) with Lemma 7, we obtain

$$\begin{aligned}&\left|\left\{(x,y)\in\mathbb{T}^2:\max_{2\le l\le\nu}\left|\sum_{j=2}^{l}T_n(x,y)\right|>\frac{\sqrt{m}}{240}\right\}\right| \\ &>\left|\left\{(x,y)\in\mathbb{T}^2:\max_{2\le l\le\nu}\left|\sum_{j=2}^{l}a_{\sigma(n)}(x,y)\right|>\frac{\sqrt{m}}{120}\right\}\right|>1.\end{aligned}$$

and therefore we will have (2.53). □

Lemma 10. *For any $\delta>0$ and $m,s\in\mathbb{N}$ there exist a sequence of regions Δ_n of the form (1.4) and polynomials Q_n, $n=2,3,\cdots,\nu=2^m$,* *such that*

$$\rho(\Delta_n) < 1+\delta, \quad n = 2,3,\cdots,\nu, \tag{2.61}$$

$$\Delta_1 = \Delta(s,s), \quad \Delta_n \subset \Delta_{n+1}, \quad n = 1,2,,\cdots,\nu-1, \tag{2.62}$$

$$\operatorname{spec} Q_n \subset (\Delta_n \setminus \Delta_{n-1}) \cap \mathbb{R}^2_+, \quad n = 2,3,,\cdots,\nu, \tag{2.63}$$

$$\left\| \sum_{j=2}^{\nu} Q_j \right\|_\infty \le c_1, \tag{2.64}$$

$$\left| \left\{ (x,y) \in \mathbb{T}^2 : \max_{2\le n\le \nu} \left| \sum_{j=2}^{n} Q_j(x,y) \right| > c_2\sqrt{m} \right\} \right| > 1. \tag{2.65}$$

Proof. Consider the sectors

$$V_n = V\left(\frac{3\pi}{4} + \frac{\varepsilon}{n}, \pi\right) \subset (-\infty, 0] \times [0,+\infty), \quad n = 2,3,\cdots,\nu,$$

of the form (2.50) and set $S_n = V_n \setminus V_{n-1}$. Applying Lemma 9, we find polynomials $T_n(x,y)$ with the properties (2.51)-(2.53). We have $\operatorname{spec}(T_n) \subset [-l,0] \times [0,l]$, $n = 2,3,\cdots,\nu$, for some integer $l > s$. Denote

$$\begin{aligned} &Q_n(x,y) = T_n(x,y)e^{ilx}, \\ &\Theta_n = ((l,0)+V_n) \cap \mathbb{R}^2_+, \quad n = 2,3,\cdots,\nu. \end{aligned}$$

Observe that Θ_n are triangles of the form (1.5), and the regions

$$\begin{aligned} \Delta_n = \Theta_n &\cup \{(x,y) \in \mathbb{R}^2 : (-x,y) \in \Theta_n\} \\ &\cup \{(x,y) \in \mathbb{R}^2 : (-x,-y) \in \Theta_n\} \\ &\cup \{(x,y) \in \mathbb{R}^2 : (x,-y) \in \Theta_n\} \end{aligned}$$

have the form (1.4). It is clear that small enough number ε guarantees (2.61). The conditions (2.62) and (2.63) are immediate. The relation (2.64) follows from (2.52), (2.65) follows from (2.53), since we have

$$\left| \sum_{j=2}^{n} Q_j(x,y) \right| = \left| \sum_{j=2}^{n} T_j(x,y) \right|.$$

Lemma is proved. □

The following lemma is a version of Lemma 10 for balls instead of triangles. It will be used in the proof of Theorem 2.

Lemma 11. *For any $\delta > 0$ and $m, r \in \mathbb{N}$ there exist a sequence of balls U_n and polynomials Q_n, $n = 2, 3, \cdots, \nu = 2^m$, such that*

$$\tau(U_n) < \delta, \quad n = 2, 3, \cdots, \nu, \tag{2.66}$$

$$B(0, 0, r) \subset U_n, \quad n = 2, 3, , \cdots, \nu, \tag{2.67}$$

$$\operatorname{spec} Q_n \subset \left(U_n \setminus \bigcup_{k=2}^{n-1} U_k \right), \quad n = 2, 3, , \cdots, \nu, \tag{2.68}$$

$$\left\| \sum_{j=2}^{\nu} Q_j \right\|_\infty \le c_1, \tag{2.69}$$

$$\left| \left\{ (x, y) \in \mathbb{T}^2 : \max_{2 \le n \le \nu} \left| \sum_{j=2}^{n} Q_j(x, y) \right| > c_2 \sqrt{m} \right\} \right| > 1. \tag{2.70}$$

Proof. Consider the sectors

$$V_n = V\left(\theta_{2n+1}, \theta_{2n}\right), \quad n = 2, 3, \cdots, \nu,$$

where

$$\theta_k = \frac{\pi}{2} + \frac{\varepsilon}{k}.$$

Applying Lemma 9, we find polynomials $T_n(x, y)$ with the properties (2.51)-(2.53). Denote

$$Q_n(x, y) = T_n(x, y) e^{iRx},$$
$$\Theta_n = (R, 0) + V_n, \quad n = 2, 3, \cdots, \nu,$$

where the number R will be determined bellow. We have

$$\operatorname{spec}(Q_n) \subset \Theta_n.$$

Consider the balls

$$B_n = B\left(0, -R\tan(\varepsilon/n), R/\cos(\varepsilon/n)\right)$$

and the lines L_n given by the formulae

$$x = R + t\cos(\theta_n), \quad y = t\sin(\theta_n).$$

Note that the boundary of the sector Θ_n is determined by the lines L_{2n} and L_{2n-1}. Besides, the line L_n is tangential for the ball B_n at the point $(R, 0)$. A simple calculation shows that

$$\tau(B_n) = \sin(\varepsilon/n) < \sin\varepsilon. \tag{2.71}$$

Using this, for a bigger enough R we will have a good approximation of the lines L_n by the balls B_n and therefore we will get

$$\operatorname{spec}(Q_n) \subset B_k, \quad k > 2n, \tag{2.72}$$

$$\operatorname{spec}(Q_n) \cap B_k = \varnothing, \quad k \le 2n. \tag{2.73}$$

We denote $U_n = B_{2n}$. A small enough number ε guarantees (2.66) according to (2.71). Then taking R bigger enough we derive (2.67). It easy to check that the relation (2.68) can be obtained from (2.72) and (2.73). Then (2.69) follows from (2.52), (2.70) follows from (2.53), since we have

$$\left|\sum_{j=2}^{n} Q_j(x,y)\right| = \left|\sum_{j=2}^{n} T_j(x,y)\right|.$$

Lemma is proved. □

3. Proof of theorems

Proof of Theorem 1. Fix integers μ_k, $k = 1, 2, \dots$ satisfying

$$\mu_{k+1} - \mu_k = 2^{k^6} - 1, \quad k = 1, 2, \dots. \tag{3.1}$$

Applying Lemma 10 successively, we may define polynomials $Q_n(x,y)$ and regions Δ_n, $n = 1, 2, \dots$, of the form (1.4) satisfying the relations

$$\rho(\Delta_n) < 1 + 1/n, \quad n = 1, 2, \cdots, \tag{3.2}$$

$$\Delta_{n-1} \subset \Delta_n, \quad n = 1, 2, \cdots, \tag{3.3}$$

$$\operatorname{spec} Q_n \subset (\Delta_n \setminus \Delta_{n-1}) \cap \mathbb{R}_+^2, \quad n = 1, 2, \cdots, \tag{3.4}$$

$$\left\|\sum_{j=\mu_k+1}^{\mu_{k+1}} Q_j\right\|_\infty < c_1, \quad k = 1, 2, \dots, \tag{3.5}$$

$$\left|\left\{(x,y) \in \mathbb{T}^2 : \max_{\mu_k < n \le \mu_{k+1}} \left|\sum_{j=\mu_k+1}^{n} Q_j(x,y)\right| > c_2 k^3\right\}\right| > 1. \tag{3.6}$$

It is clear that we may define $g_n(x,y)$ equal to one of the following real polynomials

$$\operatorname{Re}(Q_n(x,y)), \quad \operatorname{Im}(Q_n(x,y)),$$

such that

$$\left|\left\{(x,y) \in \mathbb{T}^2 : \max_{\mu_k < n \le \mu_{k+1}} \left|\sum_{j=\mu_k+1}^{n} g_j(x,y)\right| > \frac{c_2 k^3}{2}\right\}\right| > \frac{1}{2}. \tag{3.7}$$

Each $g_n(x,y)$ can be considered as a complex polynomial and from (3.4) it follows that

$$\operatorname{spec} g_n \subset \Delta_n \setminus \Delta_{n-1}, \quad n = 1, 2, \dots.$$

Consider a real function

$$f(x,y) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=\mu_k+1}^{\mu_{k+1}} g_j(n_k x, n_k y), \tag{3.8}$$

where $n_k \nearrow \infty$ is a sequence of integers that will be defined bellow. Since this series converges uniformly, f is continuous. We denote

$$E_k = \left\{ (x,y) \in \mathbb{T}^2 : \max_{\mu_k < n \le \mu_{k+1}} \left| \sum_{j=\mu_k+1}^{n} g_j(x,y) \right| > \frac{c_2 k^3}{2} \right\},$$
$$\tilde{\Delta}_j = \{(u,v) \in \mathbb{R}^2 : (u/n_k, v/n_k) \in \Delta_j\}, \quad \mu_k < j \le \mu_{k+1}.$$

It is clear that each $\tilde{\Delta}_j$ is also a region of the form (1.4) and from (3.2)-(3.4), (3.7) we get respectively

$$\rho(\tilde{\Delta}_n) < 1/n, \quad n = 1, 2, \cdots, \tag{3.9}$$
$$\tilde{\Delta}_{n-1} \subset \tilde{\Delta}_n, \quad n = 1, 2, \cdots, \tag{3.10}$$
$$\operatorname{spec}\big(g_n(n_k x, n_k y)\big) \subset \tilde{\Delta}_n \setminus \tilde{\Delta}_{n-1},\ \mu_k < n \le \mu_{k+1}, \tag{3.11}$$
$$|E_k| > 1/2. \tag{3.12}$$

According to Lemma 2, we may define the integers n_k such that

$$|\cap_{l \ge 1} \cup_{k \ge l} E_k(n_k)| = 4\pi^2. \tag{3.13}$$

It is clear that if $(x,y) \in E_k(n_k)$, then

$$\max_{\mu_k < n \le \mu_{k+1}} \left| \sum_{j=\mu_k+1}^{n} g_j(n_k x, n_k y) \right| > \frac{c_2 k^3}{2}. \tag{3.14}$$

From (3.10) and (3.11) it follows that

$$\left| S_{\tilde{\Delta}_n}(x,y,f) - S_{\tilde{\Delta}_{\mu_k}}(x,y,f) \right| = \frac{1}{k^2} \left| \sum_{j=\mu_k+1}^{n} g_j(n_k x, n_k y) \right|$$

for any $\mu_k < n \le \mu_{k+1}$. Combining this with (3.14), we get

$$\max_{\mu_k < n \le \mu_{k+1}} \left| S_{\tilde{\Delta}_n}(x,y,f) - S_{\tilde{\Delta}_{\mu_k}}(x,y,f) \right| > \frac{c_2 k}{2}, \quad (x,y) \in E_k(n_k),$$

then taking into account of (3.13), we find that the sums $S_{\tilde{\Delta}_n}(x,y,f)$ diverge almost everywhere. This proves Theorem 1. □

In the proof of Theorem 2 we use the same argument as in Theorem 1. So the details of the proof will be omitted.

Proof of Theorem 2. Fix integers μ_k, $k = 1, 2, \ldots$, satisfying (3.1). Applying Lemma 10, we define polynomials $Q_n(x,y)$ and balls B_n, $n =$

1, 2, . . ., satisfying the relations

$$\tau(U_n) < 1/n, \quad n = 2, 3, \cdots, \nu,$$

$$\operatorname{spec} Q_n \subset \left(U_n \setminus \bigcup_{k=2}^{n-1} U_k\right), \quad n = 1, 2, \cdots,$$

$$\left\| \sum_{j=\mu_k+1}^{\mu_{k+1}} Q_j \right\|_\infty < c_1, \quad k = 1, 2, \ldots,$$

$$\left| \left\{ (x, y) \in \mathbb{T}^2 : \max_{\mu_k < n \le \mu_{k+1}} \left| \sum_{j=\mu_k+1}^{n} Q_j(x, y) \right| > c_2 k^3 \right\} \right| > 1.$$

Then the function

$$f(x, y) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=\mu_k+1}^{\mu_{k+1}} Q_j(n_k x, n_k y),$$

where n_k are properly defined integers, satisfies (1.7). The rest part of the proof is exactly the same as in the proof of Theorem 1, so we leave it to our patient reader. □

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