ALMOST EVERYWHERE STRONG SUMMABILITY OF MARCINKIEWICZ MEANS OF DOUBLE WALSH-FOURIER SERIES

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ABSTRACT. In this paper we study the a. e. strong convergence of the quadratical partial sums of the two-dimensional Walsh-Fourier series. Namely, we prove the a.e. relation $(\frac{1}{n}\sum_{m=0}^{n-1}|S_{mm}f-f|^{q})^{1/q}\to 0$ for every two-dimensional functions belonging to $L\log L$ and $q>0$. From the theorem of Getsadze [7] it follows that the space $L\log L$ can not be enlarged with preserving this strong summability property.

1. Introduction

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N}:=\mathbb{P}\cup\{0\}$. Denote by $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is $\mathbb{Z}_2=\{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups $\mathbb{Z}_2$. The elements of G are of the form $x=(x_0,x_1,...,x_k,...)$ with $x_k\in\{0,1\}$ $(k\in\mathbb{N})$. The group operation on G is the coordinate-wise addition, the measure (denote by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$\begin{aligned} I_0(x) &: = G,\ \ I_n(x):=\ I_n(x_0,...,x_{n-1}) \\ &: = \{y\in G:\ y=(x_0,...,x_{n-1},y_n,y_{n+1},...)\}, \end{aligned}$$

$$(x\in G,n\in\mathbb{N}).$$

These sets are called the dyadic intervals. Let $0=(0:i\in\mathbb{N})\in G$ denote the null element of G, $I_n:=I_n(0)$ $(n\in\mathbb{N})$. Set $e_n:=(0,...,0,1,0,...)\in G$ the n th coordinate of which is 1 and the rest are zeros $(n\in\mathbb{N})$.

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For $k\in\mathbb{N}$ and $x\in G$ denote by

$$r_k(x):=(-1)^{x_k}\quad (x\in G, k\in\mathbb{N})$$

the k-th Rademacher function. If $n\in\mathbb{N}$, then $n=\sum_{i=0}^{\infty}n_i2^i$, where $n_i\in\{0,1\}$ $(i\in\mathbb{N})$, i. e. n is expressed in the number system of base 2. For $n>0$ denote $|n|:=\max\{j\in\mathbb{N}:n_j\neq 0\}$, that is, $2^{|n|}\leq n<2^{|n|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x):=\prod_{k=0}^{\infty}(r_k(x))^{n_k}=(-1)^{\sum_{k=0}^{|n|}n_kx_k}\quad (x\in G, n\in\mathbb{P}),$$

and $w_0:=1$. The Walsh-Dirichlet kernel is defined by

$$D_n(x)=\sum_{k=0}^{n-1}w_k(x).$$

Recall that (see [15] and [35])

$$D_{2^n}(x)=\begin{cases}2^n, \text{if } x\in I_n,\\ 0,\ \text{ if } x\in \overline{I}_n.\end{cases}\tag{1}$$

We consider the double system $\{w_n(x)\times w_m(y):\ n,m\in\mathbb{N}\}$ on the $G\times G$. The notiation $a\lesssim b$ in the whole paper stands for $a\leq c\cdot b$, where c is a constant depended on the q.

The rectangular partial sums of the 2-dimensional Walsh-Fourier series are defined as follows:

$$S_{M,N}(x,y,f):=\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\widehat{f}(i,j)\,w_i(x)\,w_j(y),$$

where the number

$$\widehat{f}(i,j)=\int_{G\times G}f(x,y)\,w_i(x)\,w_j(y)\,d\mu(x,y)$$

is said to be the (i,j)th Walsh-Fourier coefficient of the function f.

Denote

$$S_n^{(1)}(x,y,f):=\sum_{l=0}^{n-1}\widehat{f}(l,y)\,w_l(x),$$

$$S_m^{(2)}(x,y,f):=\sum_{r=0}^{m-1}\widehat{f}(x,r)\,w_r(y),$$

where

$$\widehat{f}(l,y)=\int_G f(x,y)\,w_l(x)\,d\mu(x)$$

and

$$\widehat{f}(x,r)=\int\limits_{G} f(x,y)\,w_r(y)\,d\mu(y).$$

The norm (or quasinorm) of the space $L_p(G\times G)$ is defined by

$$\|f\|_p:=\left(\int\limits_{G\times G}|f(x,y)|^p\,d\mu(x,y)\right)^{1/p}\qquad(0<p<+\infty).$$

We denote by $L\log L(G\times G)$ the class of measurable functions f, with

$$\int\limits_{G\times G}|f|\log^+|f|<\infty$$

where $\log^+ u:=\mathbb{I}_{(1,\infty)}\log u$ and $\mathbb{I}_E$ is character function of the set E.

Denote by $S_n^T(x,f)$ the partial sums of the trigonometric Fourier series of f and let

$$\sigma_n^T(x,f)=\frac{1}{n+1}\sum_{k=0}^{n}S_k^T(x,f)$$

be the $(C,1)$ means. Fejér [1] proved that $\sigma_n^T(f)$ converges to f uniformly for any 2π-periodic continuous function. Lebesgue in [19] established almost everywhere convergence of $(C,1)$ means if $f\in L_1(\mathbb{T}),\mathbb{T}:=[-\pi,\pi)$. The strong summability problem, i.e. the convergence of the strong means

$$\frac{1}{n+1}\sum_{k=0}^{n}\left|S_k^T(x,f)-f(x)\right|^p,\quad x\in\mathbb{T},\quad p>0,\tag{2}$$

was first considered by Hardy and Littlewood in [16]. They showed that for any $f\in L_r(\mathbb{T})$ $(1<r<\infty)$ the strong means tend to 0 a.e., if $n\to\infty$. The Fourier series of $f\in L_1(\mathbb{T})$ is said to be (H,p)-summable at $x\in T$, if the values (2) converge to 0 as $n\to\infty$. The (H,p)-summability problem in $L_1(\mathbb{T})$ has been investigated by Marcinkiewicz [24] for $p=2$, and later by Zygmund [44] for the general case $1\le p<\infty$. Oskolkov in [26] proved the following: Let $f\in L_1(\mathbb{T})$ and let Φ be a continuous positive convex function on $[0,+\infty)$ with $\Phi(0)=0$ and

$$\ln\Phi(t)=O(t/\ln\ln t)\qquad(t\to\infty).\tag{3}$$

Then for almost all x

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\Phi\left(\left|S_k^T(x,f)-f(x)\right|\right)=0.\tag{4}$$

It was noted in [26] that Totik announced the conjecture that (4) holds almost everywhere for any $f\in L_1(\mathbb{T})$, provided

$$\ln\Phi(t)=O(t)\quad(t\to\infty).\tag{5}$$

In [27] Rodin proved

Theorem A. *Let $f \in L_1(\mathbb{T})$. Then for any $A > 0$*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left(\exp\left(A\left|S_k^T(x,f) - f(x)\right|\right) - 1\right) = 0$$

for a. e. $x \in \mathbb{T}$.

Karagulyan [17] proved that the following is true.

Theorem B. *Suppose that a continuous increasing function $\Phi : [0,\infty) \to [0,\infty), \Phi(0) = 0$, satisfies the condition*

$$\limsup_{t\to+\infty} \frac{\log\Phi(t)}{t} = \infty. \tag{6}$$

Then there exists a function $f \in L_1(\mathbb{T})$ for which the relation

$$\limsup_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi\left(\left|S_k^T(x,f)\right|\right) = \infty$$

holds everywhere on $\mathbb{T}$.

For quadratic partial sums of two-dimensional trigonometric Fourier series Marcinkiewicz [25] has proved, that if $f \in L\log L\left(\mathbb{T}^2\right), \mathbb{T} := [-\pi,\pi)^2$, then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left(S_{kk}^T(x,y,f) - f(x,y)\right) = 0$$

for a. e. $(x,y) \in \mathbb{T}^2$. Zhizhiashvili [42] improved this result showing that class $L\log L\left(\mathbb{T}^2\right)$ can be replaced by $L_1\left(\mathbb{T}^2\right)$.

From a result of Konyagin [18] it follows that for every $\varepsilon > 0$ there exists a function $f \in L\log^{1-\varepsilon}\left(\mathbb{T}^2\right)$ such that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left|S_{kk}^T(x,y,f) - f(x,y)\right| \neq 0 \quad \text{for a. e. } (x,y) \in \mathbb{T}^2. \tag{7}$$

These results show that in the case of one dimensional functions the $(C,1)$ summability and $(C,1)$ strong summability we have the same maximal convergence spaces. That is, in both cases we have L_1. But, the situation changes as we step further to the case of two dimensional functions. In other words, the spaces of functions with almost everywhere summable Marcinkiewicz and strong Marcinkiewicz means are different.

The results on strong summation and approximation of trigonometric Fourier series have been extended for several other orthogonal systems. For instance, concerning the Walsh system see Schipp [31, 32, 33], Fridli [2, 3], Leindler [19, 20, 21, 22, 23], Totik [36, 37, 38], Fridli and Schipp [3], Rodin [28], Weisz [40, 41], Gabisonia [4].

The problems of summability of cubical partial sums of multiple Fourier series have been investigated by Gogoladze [12, 13, 14], Wang [39], Zhag [43], Glukhov [8], Goginava [9], Gát, Goginava, Tkebuchava [5], Goginava, Gogoladze [10], Goginava, Gogoladze, Karagulyan [11] .

Almost everywhere H^p-summability of Walsh-Fourier series with $p > 0$ proved by F. Schipp in [32]. Almost everywhere Φ-summability with condition proved by V. Rodin [29].

Theorem C (Rodin)**.** *If $\Phi(t) : [0, \infty) \to [0, \infty)$, $\Phi(0) = 0$, is an increasing continuous function satisfying (5), then the partial sums of Walsh-Fourier series of any function $f \in L_1(G)$ satisfy the condition*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Phi\left(|S_k(x, f) - f(x)|\right) = 0$$

almost everywhere on G.

In [11] we established, that, as in trigonometric case [17], the bound (5) is sharp for a.e. Φ-summability of Walsh-Fourier series. Moreover, we proved

Theorem D. *If an increasing function $\Phi(t) : [0, \infty) \to [0, \infty)$ satisfies the condition (6), then there exists a function $f \in L_1(G)$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Phi\left(|S_k(x, f)|\right) = \infty$$

holds everywhere on $[0, 1)$.

In [30] Schipp investigated the strong (H, p)- and BMO - summability of Walsh-Fourier series. Among others he gave a characterization of points in which the Walsh-Fourier series of an integrable function is (H, p)- and BMO-summable. This is the analogue of Gabisonia's result that characterizes the points of strong summability with respect to the trigonometric system.

In [10] it is studied the exponential uniform strong approximation of the Marcinkiewicz means of the two-dimensional Walsh-Fourier series. We say that the function ψ belongs to the class Ψ if it increases on $[0, +\infty)$ and

$$\lim_{u\to 0} \psi(u) = \psi(0) = 0.$$

Theorem E ([10])**.** *a)Let $\varphi \in \Psi$ and let the inequality*

$$\overline{\lim_{u\to\infty}} \frac{\varphi(u)}{\sqrt{u}} < \infty$$

holds. Then for any function $f \in C(G \times G)$ the equality

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{l=1}^{n} \left(e^{\varphi(|S_{ll}(f) - f|)} - 1 \right) \right\|_C = 0$$

is satisfied.

b) For any function $\varphi \in \Psi$ satisfying the condition

$$\overline{\lim_{u\to\infty}} \frac{\varphi(u)}{\sqrt{u}} = \infty$$

there exists a function $F \in C(G \times G)$ *such that*

$$\overline{\lim_{m \to \infty}} \frac{1}{m} \sum_{l=1}^{m} \left(e^{\varphi(|S_{ll}(0,0,F) - F(0,0)|)} - 1 \right) = +\infty.$$

For the two-dimensional Walsh-Fourier series Weisz [41] proved that if $f \in L_1(G \times G)$, then

$$\frac{1}{n} \sum_{j=0}^{n-1} \left(S_{j,j}(x, y; f) - f(x, y) \right) \to 0$$

for a. e. $(x, y) \in G \times G$.

In this paper we consider the strong means

$$H_n^q f := \left(\frac{1}{2^n} \sum_{m=0}^{2^n - 1} |S_{mm} f|^q \right)^{1/q}$$

and the maximal strong operator

$$H_*^q f := \sup_{n \in \mathbb{N}} H_n^q f.$$

We study the a. e. convergence of strong Marcinkiewicz means of the two-dimensional Walsh-Fourier series. In particular, the following is true.

Theorem 1. *Let* $f \in L \log L (G \times G)$ *and* $q > 0$*. Then*

$$\mu \{ H_*^q f > \lambda \} \lesssim \frac{1}{\lambda} \left(1 + \iint_{G \times G} |f| \log^+ |f| \right).$$

The weak type $\left(L \log^+ L, 1 \right)$ inequality and the usual density argument of Marcinkiewicz and Zygmund imply

Theorem 2. *Let* $f \in L \log L (G \times G)$ *and* $q > 0$*. Then*

$$\left(\frac{1}{n} \sum_{m=0}^{n-1} |S_{mm}(x, y, f) - f(x, y)|^q \right)^{1/q} \to 0 \text{ for a.e. } (x, y) \in G \times G \text{ as } n \to \infty.$$

We note that from the theorem of Getsadze [7] it follows that the class $L \log L$ in the last theorem is necessary in the context of strong summability question. That is, it is not possible to give a larger convergence space (of the form $L \log L \phi(L)$ with $\phi(\infty) = 0$) than $L \log L$. This means a sharp contrast between the one and two dimensional strong summability.

We also note that in the case of trigonometric system Sjölin proved [34] that for every $r > 1$ and two variable function $f \in L_r(\mathbb{T}^2)$ the almost everywhere convergence $S_{nn} f \to f$ $(n \to \infty)$ holds. Since this issue with respect to the Walsh system is still open, then in this point of view Theorem 2 may seem more interesting.

2. The maximal operators of the Walsh-Gabisonia operators

Let $1 < p \leq 2, 1/p + 1/q = 1, q \in \mathbb{N}$ and let f be an integrable function. In [30] it is introduced the notion of the dyadic Walsh-Gabisonia operator in the following way

$$V_n^{(p)} f(x) = \left(2^{-n(p-1)} \int_G \left| \sum_{k=0}^{n} 2^k \mathbb{I}_{I_k}(u) E_n f(x+u+e_k) \right|^p d\mu(u) \right)^{1/p},$$

where $E_n f := S_{2^n}(f)$. We prove the following lemma with respect to the maximal Walsh-Gabisonia operators $V^{(p)} f := \sup_n |V_n^{(p)} f|$.

Lemma 1. *Let $f \in L_1(G)$ and $1 < p \leq 2$. Then for each $\lambda > 0$ we have*

$$\mu\left(\left\{V^{(p)} f > \lambda\right\}\right) \lesssim \|f\|_1 / \lambda.$$

In order to prove Lemma 1 we shall use the Calderon-Zygmund decomposition in the following form ([35]).

Lemma 2. *Calderon-Zygmund lemma. Let $f \in L_1$ and $\lambda > \|f\|_1$. Then there exist a sequence of pairwise disjoint intervals $J_k \subset G (k \in \mathbb{P})$ and a decomposition $f = \sum_{k=0}^{\infty} f_k$ of the function f such that:*

$$\|f_0\|_\infty \leq 2\lambda, \quad \{f_k \neq 0\} \subset J_k, \quad \int_{J_k} f_k(s) d\mu(s) = 0, \quad 2^{-\nu_k} = \mu(J_k),$$

$$2^{\nu_k} \int_{J_k} |f_k(s)| ds \leq 4\lambda \ (k \in \mathbb{P}), \quad \sum_{k=1}^{\infty} 2^{-\nu_k} \leq \frac{1}{\lambda} \int_U |f(s)| d\mu(s),$$

$$U := \bigcup_{k=1}^{\infty} J_k = \{x \in G : (E^*|f|)(x) > \lambda\},$$

where $E^* f = \sup_n |E_n f|$.

Proof of Lemma 1. It can be supposed that $p < 2$, since for the case $p = 2$ Lemma 1 is proved by Schipp in [30]. It is easy to get that $p = \frac{q}{q-1}, p - 1 = \frac{1}{q-1}$. We will heavily use the fact that q is an integer.

Since we have to give an upper bound for $\mu\left(\left\{x : V^{(p)} f(x) > \lambda\right\}\right)$ and we have $\mu(U) \leq \frac{1}{\lambda}\|f\|_1$, the in the sequel we suppose that $x \notin U$. We give some bound for $V_n^{(p)} f(x)$ such that it will be independent from n and consequently it also will be an upper bound for $V^{(p)} f(x)$. By Lemma 2 we have for $\nu_j \geq n$ that $E_n f_j \equiv 0$. Thus, $\nu_j < n$ is supposed everywhere in the proof of this lemma. Besides, if $n > k \geq \nu_j$, then $E_n f_j(x + e_k + u) = 0$, since $u \in I_k$ and thus $u + e_k \in I_k$ and $x \notin J_j = I_{\nu_j}(s)$ (for some $s \in G$). And $\nu_j \leq k$ would give $x + u + e_k \notin I_{\nu_j}(s)$ and consequently $f_j(x + e_k + u + v) = 0$ for every $v \in I_n$. This really gives $E_n f_j(x + e_k + u) = 0$ for $x \in \bar{U}$ and $n \geq k \geq \nu_j$. That is, $n > \nu_j > k$ can be supposed.

Moreover,

$$E_n f_j(x+e_k+u) = E_n(f_j \mathbb{I}_{J_j})(x+e_k+u)$$
$$= \mathbb{I}_{J_j}(x+e_k+u) E_n f_j(x+e_k+u).$$

We can write

$$\left|V_n^{(p)} f(x)\right|^p$$
$$= 2^{-n(p-1)} \int_G \left| \sum_{k=0}^{n-1} 2^k \mathbb{I}_{[0,2^{-k})}(u) E_n \left(\sum_{j:\nu_j<n} f_j \right)(x+e_k+u) \right|^{\frac{q}{q-1}} d\mu(u)$$
$$\leq 2^{-n(p-1)} \int_G \left(\sum_{(j,k)\in A^{(n)}} 2^{k_1+\cdots+k_q} \mathbb{I}_{I_{k_1}}(u) \right.$$
$$\left. \times \cdots \times \mathbb{I}_{I_{k_q}}(u) \left| \prod_{i=1}^{q} E_n f_{j_i}(x+e_{k_i}+u) \right| \right)^{\frac{1}{q-1}} d\mu(u)$$
$$\leq \sum_{(j,k)\in A^{(n)}} 2^{-n(p-1)} 2^{\frac{k_1+\ldots k_q}{q-1}} \int_G \mathbb{I}_{I_{k_1}}(u)$$
$$\times \cdots \times \mathbb{I}_{I_{k_q}}(u) \prod_{i=1}^{q} |E_n f_{j_i}(x+e_{k_i}+u)|^{\frac{1}{q-1}} du,$$

where

$$A^{(n)} = \{(j,k) : j=(j_1,\ldots,j_q), k=(k_1,\ldots,k_q),$$
$$0 \leq \nu_{j_1},\ldots,\nu_{j_q} < n, 0 \leq k_1 < \nu_{j_1},\ldots, 0 \leq k_q < \nu_{j_q}\}.$$

Till the end of the proof of this lemma let ν_{j_q} be the maximum of $\nu_{j_1},\ldots,\nu_{j_q}$. For $s \in \bar{J}_{j_q}$ we have $E_n f_{j_q}(s) = 0$ and for $s \in J_{j_q}$ we have $2^{-n} E_n |f_{j_q}|(s) \leq 4\lambda |J_{j_q}|$. That is, by

$$h_{j_q}(s) := \begin{cases} 0, & s \notin J_{j_q}, \\ |J_{j_q}|^{\frac{1}{q-1}} = 2^{\frac{-\nu_{j_q}}{q-1}}, & s \in J_{j_q} \end{cases}$$

we have

$$(E_n |f_{j_q}|)^{\frac{1}{q-1}} \leq (4\lambda)^{\frac{1}{q-1}} 2^{\frac{n}{q-1}} h_{j_q}.$$

Thus, from above written it follows that for $x \in \bar{U} = \overline{\bigcup_{i=1}^{\infty} J_i}$ we have

$$\left|V_n^{(p)} f(x)\right|^p \leq \sum_{(j,k)\in A^{(n)}} 2^{-n(p-1)} 2^{\frac{k_1+\cdots+k_q}{q-1}}$$
$$\times \int_G \mathbb{I}_{I_{k_1}}(u) \times \cdots \times \mathbb{I}_{I_{k_q}}(u)$$

$$\times \prod_{i=1}^{q-1} |E_n f_{j_i}(x+e_{k_i}+u)|^{\frac{1}{q-1}} (4\lambda)^{p-1} 2^{n(p-1)} h_{j_q}(x+e_{j_q}+u) d\mu(u).$$

By the inequality of the geometric and arithmetic means we have

$$\prod_{i=1}^{q-1} |E_n f_{j_i}(x+e_{k_i}+u)|^{\frac{1}{q-1}}$$

$$\leq \frac{1}{q-1} \sum_{i=1}^{q-1} |E_n f_{j_i}(x+e_{k_i}+u)| \prod_{i=1}^{q-1} \mathbb{I}_{J_{j_i}}(x+e_{k_i}+u).$$

Set

$$A_1^{(n)} = \left\{(j,k) : j=(j_1,\ldots,j_q), k=(k_1,\ldots,k_q), 0 \leq \nu_{j_1},\ldots,\nu_{j_q} < n,\right.$$

$$\left. 0 \leq k_1 < \nu_{j_1},\ldots, 0 \leq k_q < \nu_{j_q}, \nu_{j_q} = \max\left\{\nu_{j_1},\ldots,\nu_{j_q}\right\}\right\}.$$

This by the fact that $h_{j_q} = 2^{-\frac{\nu_{j_q}}{q-1}} \cdot \mathbb{I}_{J_{j_q}}$ follows for $x \in \bar{U}$

$$\left|V_n^{(p)} f(x)\right|^p$$

$$\lesssim (4\lambda)^{p-1} \sum_{i=1}^{q-1} \sum_{j_i=1}^{\infty} \sum_{(j,k)\in A_1^n} 2^{\frac{k_1+\cdots+k_q}{q-1}} 2^{-\frac{\nu_{j_q}}{q-1}}$$

$$\times \int_G \mathbb{I}_{I_{k_1}}(u) \times \cdots \times \mathbb{I}_{I_{k_q}}(u) \, |E_n f_{j_i}(x+e_{k_i}+u)|$$

$$\times \prod_{l=1}^{q} \mathbb{I}_{J_{j_l}}(x+e_{k_l}+u) d\mu(u)$$

$$\lesssim (4\lambda)^{p-1} \sum_{i=1}^{q-1} \sum_{j_i=1}^{\infty} \sum_{(j,k)\in A_1^n} 2^{\frac{k_1+\cdots+k_q}{q-1}} 2^{-\frac{\nu_{j_q}}{q-1}}$$

$$\times \int_G \mathbb{I}_{I_{k_1}}(u) \times \cdots \times \mathbb{I}_{I_{k_q}}(u) \, |f_{j_i}(x+e_{k_i}+u)| \prod_{l=1}^{q} \mathbb{I}_{J_{j_l}}(x+e_{k_l}+u) d\mu(u).$$

In the sum $\sum_{(j,k)\in A_1^n}$ fix the numbers $k_1,\ldots,k_q$ and the values of ν_{j_i} and ν_{j_q} (which is the biggest among $\nu_{j_1},\ldots \nu_{j_q}$). The only member in the sum above depending on $j_1,\ldots,j_{i-1},j_{i+1},\ldots j_q$ is the product $\prod_{l=1}^{q} \mathbb{I}_{J_{j_l}}(x+e_{k_l}+u)$. Since for a fixed $l \in \{1,\ldots,i-1,i+1,\ldots,q\}$ the intervals J_{j_l} are disjoint, then $\sum_{j_l=1}^{\infty} \mathbb{I}_{J_{j_l}}(x+e_{k_l}+u) \leq \mathbb{I}_U(x+e_{k_l}+u) \leq 1$. Set

$$A_2^{n,i} := \left\{k=(k_1,\ldots,k_q) : 0 \leq k_1,\ldots,k_q < \nu_{j_q}, k_i < \nu_{j_i} \leq \nu_{j_q} < n\right\}$$

$$\subset A_2^i := \left\{ k = (k_1, \ldots, k_q) : 0 \leq k_1, \ldots, k_q < \nu_{j_q}, k_i < \nu_{j_i} \leq \nu_{j_q} \right\}$$

$$= \left\{ k \in A_2^i : \max_{1 \leq s \leq q} k_s \leq \nu_{j_i} \right\} \cup \left\{ k \in A_2^i : \max_{1 \leq s \leq q} k_s > \nu_{j_i} \right\} =: A_3^i \cup A_4^i.$$

Thus, since A_2^i already does not depend on n we have

$$\left| V_n^{(p)} f(x) \right|^p = \sup_n \left| V_n^{(p)} f(x) \right|^p$$

$$\lesssim (4\lambda)^{p-1} \sum_{i=1}^{q-1} \sum_{j_i=1}^{\infty} \sum_{\nu_{j_q}=\nu_{j_i}}^{\infty} \sum_{k \in A_2^i} 2^{\frac{k_1 + \cdots + k_q}{q-1}} 2^{-\frac{\nu_{j_q}}{q-1}}$$

$$\times \int_G \mathbb{I}_{I_{k_1}}(u) \times \cdots \times \mathbb{I}_{I_{k_q}}(u) \left| f_{j_i}(x + e_{k_i} + u) \right| d\mu(u).$$

Consequently,

$$\int_{\bar{U}} \left| V^{(p)} f(x) \right|^p dx$$

$$\lesssim (4\lambda)^{p-1} \sum_{i=1}^{q-1} \sum_{j_i=1}^{\infty} \sum_{\nu_{j_q}=\nu_{j_i}}^{\infty} \sum_{k \in A_2^i} \int_{\bar{U}} 2^{\frac{k_1 + \cdots + k_q}{q-1}} 2^{-\frac{\nu_{j_q}}{q-1}}$$

$$\times \left(\int_G \mathbb{I}_{I_{k_1}}(u) \times \cdots \times \mathbb{I}_{I_{k_q}}(u) \left| f_{j_i}(x + e_{k_i} + u) \right| d\mu(u) \right) d\mu(x).$$

The sum $\sum_{k \in A_2^i}$ will be estimated as the sum of $\sum_{k \in A_3^i}$ and $\sum_{k \in A_4^i}$. Denote k^* the maximum of $k_1, \ldots, k_q$.

$$\sum_{j_i=1}^{\infty} \sum_{\nu_{j_q}=\nu_{j_i}}^{\infty} \sum_{k \in A_3^i} \int_{\bar{U}} 2^{\frac{k_1 + \cdots + k_q}{q-1}} 2^{-\frac{\nu_{j_q}}{q-1}}$$

$$\times \int_G \mathbb{I}_{I_{k_1}}(u) \times \cdots \times \mathbb{I}_{I_{k_q}}(u) \left| f_{j_i}(x + e_{k_i} + u) \right| d\mu(u)$$

$$\lesssim \sum_{j_i=1}^{\infty} \sum_{\nu_{j_q}=\nu_{j_i}}^{\infty} \sum_{k^*=0}^{\nu_{j_i}} 2^{k^* \frac{q}{q-1}} \cdot 2^{-k^*} \cdot 2^{-\frac{\nu_{j_q}}{q-1}}$$

$$\times \int_G \int_G D_{2^{k^*}}(u) \left| f_{j_i}(x + e_{k_i} + u) \right| d\mu(u, x)$$

$$\lesssim \sum_{j_i=1}^{\infty} \sum_{\nu_{j_q}=\nu_{j_i}}^{\infty} \sum_{k^*=0}^{\nu_{j_i}} 2^{\frac{k^* - \nu_{j_q}}{q-1}} \| f_{j_i} \|_1$$

$$\lesssim \sum_{j_i=1}^{\infty} \|f_{j_i}\|_1 \lesssim \|f\|_1.$$

On the other hand, the part A_4^i can be estimated as follows. For the sake of simplicity we suppose that the maximum of $k_1, \ldots, k_q$ is k_1.

$$\sum_{j_i=1}^{\infty} \sum_{\nu_{j_q}=\nu_{j_i}}^{\infty} \sum_{k \in A_4^i} \int_{\bar{U}} 2^{\frac{k_1+\cdots+k_q}{q-1}} 2^{-\frac{\nu_{j_q}}{q-1}}$$

$$\times \left(\int_G \mathbb{I}_{I_{k_1}}(u) \times \cdots \times \mathbb{I}_{I_{k_q}}(u) \left| f_{j_i}(x + e_{k_i} + u) \right| d\mu(u) \right) d\mu(x)$$

$$\leq \sum_{j_i=1}^{\infty} \sum_{\nu_{j_q}=\nu_{j_i}}^{\infty} \sum_{k_{j_i}=0}^{\nu_{j_i}-1} \sum_{k_1=\nu_{j_i}+1}^{\nu_{j_q}-1} \sum_{k_2,\ldots,k_{i-1},k_{i+1},\ldots,k_q \leq k_1} 2^{-k_1} 2^{\frac{k_1+\cdots+k_q}{q-1}} \cdot 2^{-\frac{\nu_{j_q}}{q-1}}$$

$$\times \int_G \int_G D_{2^{k_1}}(u) \left| f_{j_i}(x + e_{k_i} + u) \right| d\mu(u, x)$$

$$\lesssim \sum_{j_i=1}^{\infty} \sum_{\nu_{j_q}=\nu_{j_i}}^{\infty} \sum_{k_{j_i}=0}^{\nu_{j_i}-1} \sum_{k_1=\nu_{j_i}+1}^{\nu_{j_q}-1} 2^{-k_1} 2^{\frac{(q-1)k_1+k_{j_i}}{q-1}} \cdot 2^{-\frac{\nu_{j_q}}{q-1}} \|f_{j_i}\|_1$$

$$\lesssim \sum_{j_i=1}^{\infty} \sum_{\nu_{j_q}=\nu_{j_i}}^{\infty} \sum_{k_{j_i}=0}^{\nu_{j_i}-1} (\nu_{j_q} - \nu_{j_i}) \cdot 2^{\frac{k_{j_i}-\nu_{j_q}}{q-1}} \|f_{j_i}\|_1$$

$$\lesssim \sum_{j_i=1}^{\infty} \sum_{\nu_{j_q}=\nu_{j_i}}^{\infty} (\nu_{j_q} - \nu_{j_i}) \cdot 2^{\frac{\nu_{j_i}-\nu_{j_q}}{q-1}} \|f_{j_i}\|_1$$

$$\lesssim \sum_{j_i=1}^{\infty} \|f_{j_i}\|_1 \lesssim \|f\|_1.$$

Finally, this implies

$$\mu\left(\left\{ |V^{(p)} f|^p > \lambda \right\}\right) \leq \mu(U) + \frac{1}{\lambda^p} \int_{\bar{U}} |V^{(p)} f|^p \lesssim \frac{\|f\|_1}{\lambda}.$$

Lemma 1 is proved. $\square$

3. Proof of Theorem 1

Let $f \in L_1(G \times G)$. Then the dyadic maximal function is given by

$$Mf(x,y) := \sup_{n \in \mathbb{N}} 2^{2n} \int\limits_{I_n \times I_n} |f(x+s, y+t)| \, d\mu(s,t).$$

For a two-dimensional integrable function f we need to introduce the following hybrid maximal functions

$$M_1 f(x,y) := \sup_{n \in \mathbb{N}} 2^n \int\limits_{I_n} |f(x+s, y)| \, d\mu(s),$$

$$M_2 f(x,y) := \sup_{n \in \mathbb{N}} 2^n \int\limits_{I_n} |f(x, y+t)| \, d\mu(t),$$

$$\begin{aligned} &V_1^{(p)}(x,y,f) \\ &:= \sup_{n \in \mathbb{N}} \left(2^{-n(p-1)} \int\limits_G \left| \sum_{j=0}^{n} 2^j \mathbb{I}_{I_j}(t) S_{2^n}^{(1)} f(x+t+e_j, y) \right|^p d\mu(t) \right)^{1/p}, \end{aligned} \tag{8}$$

$$\begin{aligned} &V_2^p(x,y,f) \\ &:= \sup_{n \in \mathbb{N}} \left(2^{-n(p-1)} \int\limits_G \left| \sum_{j=0}^{n} 2^j \mathbb{I}_{I_j}(t) S_{2^n}^{(2)} f(x, y+t+e_j) \right|^p d\mu(t) \right)^{1/p}. \end{aligned} \tag{9}$$

It is well known that for $f \in L \log^+ L$ the following estimation holds

$$\begin{aligned} &\lambda \mu \{(x,y) \in G \times G : Mf(x,y) > \lambda\} \\ &\lesssim 1 + \iint\limits_{G \times G} |f(x,y)| \log^+ |f(x,y)| \, d\mu(x,y) \end{aligned} \tag{10}$$

and for $s = 1, 2$

$$\iint\limits_{G \times G} |M_s f(x,y)| \, d\mu(x,y) \lesssim 1 + \iint\limits_{G \times G} |f(x,y)| \log^+ |f(x,y)| \, d\mu(x,y). \tag{11}$$

Set

$$\Omega := \left\{ (x,y) \in G \times G : V_1^{(p)} f(x,y) > \lambda \right\}.$$

Then by Fubini's Theorem and Lemma 1 we can write

$$\mu(\Omega) = \iint_{G\times G} \mathbb{I}_\Omega(x,y)\,d\mu(x,y) = \int_G \left(\int_G \mathbb{I}_\Omega(x,y)\,d\mu(x)\right) d\mu(y) \lesssim \frac{1}{\lambda}\int_G \left(\int_G |f(x,y)|\,d\mu(x)\right) d\mu(y) \lesssim \frac{\|f\|_1}{\lambda}. \tag{12}$$

Analogously, we can prove that

$$\mu\left\{(x,y)\in G\times G : V_2^{(p)}f(x,y) > \lambda\right\} \lesssim \frac{\|f\|_1}{\lambda}. \tag{13}$$

For Dirichlet kernel Schipp proved the following representation [30, page 622]

$$D_m(x) = \sum_{k=0}^{n-1} \mathbb{I}_{I_k\setminus I_{k+1}}(x) \sum_{j=0}^{k} \varepsilon_{kj} 2^{j-1} w_m(x+e_j) - \frac{1}{2}w_m(x) + (m+1/2)\,\mathbb{I}_{I_n}(x), \tag{14}$$

where $m < 2^n$ and

$$\varepsilon_{kj} = \begin{cases} -1, & \text{if } j = 0,1,...,k-1, \\ +1, & \text{if } j = k. \end{cases}$$

Proof of Theorem 1. First, we prove that the following estimation holds

$$\left(\frac{1}{2^n}\sum_{m=0}^{2^n-1} |S_{mm}(x,y,f)|^q\right)^{1/q} \lesssim V_2^{(p)}(x,y,M_1 f) + V_1^{(p)}(x,y,M_2 f) + Mf(x,y) + V_2^{(p)}(x,y,A) + V_1^{(p)}(x,y,A) + \|f\|_1, \tag{15}$$

where A is an integrable function of two variable on $G\times G$ function of two variable which will be defined below.

Since

$$\left(\frac{1}{2^n}\sum_{m=0}^{2^n-1} |S_{mm}(x,y,f)|^{q_1}\right)^{1/q_1} \le \left(\frac{1}{2^n}\sum_{m=0}^{2^n-1} |S_{mm}(x,y,f)|^{q_2}\right)^{1/q_2},$$

when $q_1 \le q_2$, without of generality, we can suppose that $2 < q \in \mathbb{N}$.

It is easy to show that

$$
\begin{aligned}
&\left(\sum_{m=0}^{2^n-1}\left|S_{mm}\left(x,y,f\right)\right|^q\right)^{1/q}\\
=&\left(\sum_{m=0}^{2^n-1}\left|S_{mm}\left(x,y,S_{2^n,2^n}f\right)\right|^q\right)^{1/q}\\
=&\left(\sum_{m=0}^{2^n-1}\left|\iint\limits_{G\times G}S_{2^n,2^n}\left(x+s,y+t,f\right)D_m\left(s\right)D_m\left(t\right)d\mu\left(s,t\right)\right|^q\right)^{1/q}\\
\leq&\sup_{\{\alpha_{mn}(x,y)\}}\left|\iint\limits_{G\times G}S_{2^n,2^n}\left(x+s,y+t,f\right)\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)D_m\left(s\right)D_m\left(t\right)d\mu\left(s,t\right)\right|
\end{aligned}
$$

by taking the supremum over all $\{\alpha_{mn}\left(x,y\right)\}$ for which

$$
\left(\sum_{m=0}^{2^n-1}\left|\alpha_{mn}\left(x,y\right)\right|^p\right)^{1/p}\leq 1,\frac{1}{p}+\frac{1}{q}=1.
$$

From (14) we can write

$$
\begin{aligned}
&\iint\limits_{G\times G}S_{2^n,2^n}\left(x+s,y+t,f\right)\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)D_m\left(s\right)D_m\left(t\right)d\mu\left(s,t\right) \qquad (16)\\
&=\iint\limits_{G\times G}S_{2^n,2^n}\left(x+s,y+t,f\right)\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-1}\sum_{j_1=0}^{k_1}\sum_{j_2=0}^{k_2}\mathbb{I}_{I_{k_1}\backslash I_{k_1+1}}\left(s\right)\\
&\times\mathbb{I}_{I_{k_2}\backslash I_{k_2+1}}\left(t\right)\varepsilon_{k_1j_1}\varepsilon_{k_2j_2}2^{j_1+j_2-2}\\
&\times\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(s+t+e_{j_1}+e_{j_2}\right)d\mu\left(s,t\right)\\
&-\frac{1}{2}\iint\limits_{G\times G}S_{2^n,2^n}\left(x+s,y+t,f\right)\sum_{k_1=0}^{n-1}\sum_{j_1=0}^{k_1}\mathbb{I}_{I_{k_1}\backslash I_{k_1+1}}\left(s\right)\\
&\times\varepsilon_{k_1j_1}2^{j_1-1}\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(s+t+e_{j_1}\right)d\mu\left(s,t\right)\\
&+\iint\limits_{G\times G}S_{2^n,2^n}\left(x+s,y+t,f\right)\sum_{k_1=0}^{n-1}\sum_{j_1=0}^{k_1}\mathbb{I}_{I_{k_1}\backslash I_{k_1+1}}\left(s\right)\\
&\times\varepsilon_{k_1j_1}2^{j_1-1}\mathbb{I}_{I_n}\left(t\right)\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(s+e_{j_1}\right)\left(m+1/2\right)d\mu\left(s,t\right)
\end{aligned}
$$

$$-\frac{1}{2}\iint\limits_{G\times G} S_{2^n,2^n}f\left(x+s,y+t\right)\sum_{k_2=0}^{n-1}\sum_{j_2=0}^{k_2}\mathbb{I}_{I_{k_2}\backslash I_{k_2+1}}\left(t\right)$$

$$\times\varepsilon_{k_2j_2}2^{j_2-1}\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(s+t+e_{j_2}\right)d\mu\left(s,t\right)$$

$$+\frac{1}{4}\iint\limits_{G\times G} S_{2^n,2^n}\left(x+s,y+t,f\right)\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(s+t\right)d\mu\left(s,t\right)$$

$$-\frac{1}{2}\iint\limits_{G\times G} S_{2^n,2^n}\left(x+s,y+t,f\right)\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(s\right)$$

$$\times\left(m+\frac{1}{2}\right)\mathbb{I}_{I_n}\left(t\right)d\mu\left(s,t\right)$$

$$+\iint\limits_{G\times G} S_{2^n,2^n}\left(x+s,y+t,f\right)\sum_{k_2=0}^{n-1}\sum_{j_2=0}^{k_2}\mathbb{I}_{I_{k_2}\backslash I_{k_2+1}}\left(t\right)$$

$$\times\varepsilon_{k_2j_2}2^{j_2-1}\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(t+e_{j_2}\right)$$

$$\times\left(m+\frac{1}{2}\right)\mathbb{I}_{I_n}\left(s\right)d\mu\left(s,t\right)$$

$$-\frac{1}{2}\iint\limits_{G\times G} S_{2^n,2^n}\left(x+s,y+t,f\right)\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(t\right)$$

$$\times\left(m+\frac{1}{2}\right)\mathbb{I}_{I_n}\left(s\right)d\mu\left(s,t\right)$$

$$+\iint\limits_{G\times G} S_{2^n,2^n}\left(x+s,y+t,f\right)\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)$$

$$\times\left(m+\frac{1}{2}\right)^2\mathbb{I}_{I_n}\left(s\right)\mathbb{I}_{I_n}\left(t\right)d\mu\left(s,t\right)$$

$$:=\sum_{k=1}^{9}J_k.$$

It is easy to show that

$$
\begin{aligned}
|J_9| &\lesssim \left(\sum_{m=0}^{2^n-1} |\alpha_{mn}(x,y)|^p\right)^{1/p} \\
&\quad \times 2^{2n+n/q} \iint\limits_{I_n\times I_n} |f(x+s,y+t)|\, d\mu(s,t) \\
&\lesssim 2^{n/q} Mf(x,y),
\end{aligned}
\tag{17}
$$

$$
|J_5| \lesssim 2^{n/q} \left(\sum_{m=0}^{2^n-1} |\alpha_{mn}(x,y)|^p\right)^{1/p} \|f\|_1 \lesssim 2^{n/q} \|f\|_1 .
\tag{18}
$$

$$
\begin{aligned}
|J_8| &\lesssim \iint\limits_{I_n\times G} S_{2^n,2^n}(x+s,y+t,|f|) \\
&\quad \times \left|\sum_{m=0}^{2^n-1} \alpha_{mn}(x,y)\, w_m(t)\,(m+1/2)\right| d\mu(s,t) \\
&= \iint\limits_{I_n\times G} \left(2^n \int\limits_{I_n} |f(x+s,y+t+v)|\, d\mu(v)\right) \\
&\quad \times \left|\sum_{m=0}^{2^n-1} \alpha_{mn}(x,y)\, w_m(t)\,(m+1/2)\right| d\mu(s,t) \\
&= \int\limits_{G} \left(\int\limits_{I_n} \left(2^n \int\limits_{I_n} |f(x+s,y+t+v)|\, d\mu(s)\right) d\mu(v)\right) \\
&\quad \times \left|\sum_{m=0}^{2^n-1} \alpha_{mn}(x,y)\, w_m(t)\,(m+1/2)\right| d\mu(t) \\
&\lesssim \int\limits_{G} \left(\int\limits_{I_n} M_1 f(x,y+t+v)\, d\mu(v)\right) \\
&\quad \times \left|\sum_{m=0}^{2^n-1} \alpha_{mn}(x,y)\, w_m(t)\,(m+1/2)\right| d\mu(t) \\
&\lesssim 2^{-n} \int\limits_{G} S_{2^n}^{(2)}(x,y+t,M_1 f) \\
&\quad \times \left|\sum_{m=0}^{2^n-1} \alpha_{mn}(x,y)\, w_m(t)\,(m+1/2)\right| d\mu(t)
\end{aligned}
\tag{19}
$$

$$\lesssim 2^{-n}\left(\int_G\left(S^{(2)}_{2^n}(x,y+t,M_1f)\right)^p d\mu(t)\right)^{1/p}$$

$$\times\left(\int_G\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t)\,(m+1/2)\right|^q d\mu(t)\right)^{1/q}.$$

Then from the Hausdorff-Young inequality, we have

$$\left(\int_G\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t)\,(m+1/2)\right|^q d\mu(t)\right)^{1/q} \tag{20}$$

$$\sup_{\|g\|_p\le 1}\left|\int_G g(t)\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t)\,(m+1/2)\,d\mu(t)\right|$$

$$=\sup_{\|g\|_p\le 1}\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,(m+1/2)\int_G g(t)\,w_m(t)\,d\mu(t)\right|$$

$$=\sup_{\|g\|_p\le 1}\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,(m+1/2)\,\widehat{g}(m)\right|$$

$$\lesssim \sup_{\|g\|_p\le 1}\left(\sum_{m=0}^{2^n-1}\left|\alpha_{mn}(x,y)\,(m+1/2)\right|^p\right)^{1/p}$$

$$\times\left(\sum_{m=0}^{2^n-1}\left|\widehat{g}(m)\right|^q\right)^{1/q}$$

$$\lesssim \sup_{\|g\|_p\le 1} 2^n\left(\sum_{m=0}^{2^n-1}\left|\alpha_{mn}(x,y)\right|^p\right)^{1/p}\|g\|_p\lesssim 2^n.$$

Hence

$$|J_8|\lesssim 2^{n/q}\left(\sum_{m=0}^{2^n-1}\left|\alpha_{mn}(x,y)\right|^p\right)^{1/p} V^{(p)}_2(x,y,M_1f)$$

$$\lesssim 2^{n/q}V^{(p)}_2(x,y,M_1f).$$

Analogously, we can prove that

$$|J_6|\lesssim 2^{n/q}V^{(p)}_1(x,y,M_2f). \tag{21}$$

Now, we estimate J_7. Since

$$\int_{I_n} S_{2^n,2^n}(x+s,y+t,|f|)\,d\mu(s)=2^{-n}S_{2^n,2^n}(x,y+t,|f|)$$

from (20) we can write

$$
\begin{aligned}
(22)\qquad |J_7| &\lesssim \sum_{j_2=0}^{n-1}\sum_{k_2=j_2}^{n-1} 2^{j_2-1} \iint\limits_{I_n\times\left(I_{k_2}\setminus I_{k_2+1}\right)} S_{2^n,2^n}\left(x+s,y+t,|f|\right) \\
&\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(t+e_{j_2}\right)\left(m+1/2\right)\right| d\mu\left(s,t\right) \\
&\lesssim \sum_{j_2=0}^{n-1} 2^{j_2-1} \iint\limits_{I_n\times I_{j_2}} S_{2^n,2^n}\left(x+s,y+t,|f|\right) \\
&\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(t+e_{j_2}\right)\left(m+1/2\right)\right| d\mu\left(s,t\right) \\
&\lesssim \sum_{j_2=0}^{n-1} 2^{j_2-1} \int\limits_{I_{j_2}}\left(\int\limits_{I_n} S_{2^n,2^n}\left(x+s,y+t,|f|\right)d\mu\left(s\right)\right) \\
&\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(t+e_{j_2}\right)\left(m+1/2\right)\right| d\mu\left(t\right) \\
&\lesssim 2^{-n}\sum_{j_2=0}^{n-1} 2^{j_2-1} \int\limits_{I_{j_2}} S_{2^n,2^n}\left(x,y+t,|f|\right) \\
&\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(t+e_{j_2}\right)\left(m+1/2\right)\right| d\mu\left(t\right) \\
&\lesssim 2^{-n}\sum_{j_2=0}^{n-1} 2^{j_2-1} \int\limits_{I_{j_2}} S_{2^n,2^n}\left(x,y+t+e_{j_2},|f|\right) \\
&\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(t\right)\left(m+1/2\right)\right| d\mu\left(t\right) \\
&\lesssim 2^{-n}\int\limits_{G}\sum_{j_2=0}^{n-1} 2^{j_2}\mathbb{I}_{I_{j_2}}\left(t\right) S_{2^n,2^n}\left(x,y+t+e_{j_2},|f|\right) \\
&\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(t\right)\left(m+1/2\right)\right| d\mu\left(t\right) \\
&\lesssim 2^{-n}\left(\int\limits_{G}\left(\sum_{j_2=0}^{n-1} 2^{j_2}\mathbb{I}_{I_{j_2}}\left(t\right) S_{2^n,2^n}\left(x,y+t+e_{j_2},|f|\right)\right)^{p} d\mu\left(t\right)\right)^{1/p}
\end{aligned}
$$

$$\times\left(\int\limits_G\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t)\,(m+1/2)\right|^q d\mu(t)\right)^{1/q}$$

$$\lesssim\left(\int\limits_G\left(\sum_{j_2=0}^{n-1}2^{j_2}\mathbb{I}_{I_{j_2}}(t)\,S_{2^n,2^n}(x,y+t+e_{j_2},|f|)\right)^p d\mu(t)\right)^{1/p}$$

Since

$$\begin{aligned}&S_{2^n,2^n}(x,y+t+e_{j_2},|f|)\\&=2^n\int\limits_{I_n}\left(2^n\int\limits_{I_n}|f(x+u,y+t+e_{j_2}+v)|\,d\mu(u)\right)d\mu(v)\\&\lesssim 2^n\int\limits_{I_n}M_1f(x,y+t+e_{j_2}+v)\,d\mu(v)\\&=S^{(2)}_{2^n}(x,y+t+e_{j_2},M_1f)\end{aligned}$$

from (22) we can write

$$\begin{aligned}&|J_7|\\&\lesssim\left(\int\limits_G\left(\sum_{j_2=0}^{n-1}2^{j_2}\mathbb{I}_{I_{j_2}}(t)\,S^{(2)}_{2^n}(x,y+t+e_{j_2},M_1f)\right)^p d\mu(t)\right)^{1/p}\\&\lesssim 2^{n/q}V_2^{(p)}(x,y,M_1f).\end{aligned}\tag{23}$$

Analogously, we can prove that

$$|J_3|\lesssim 2^{n/q}V_1^{(p)}(x,y,M_2f).\tag{24}$$

For J_1 we can write

$$\begin{aligned}J_1&\lesssim\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-1}\sum_{j_1=0}^{k_1}\sum_{j_2=0}^{k_2}2^{j_1+j_2-2}\\&\times\iint\limits_{(I_{k_1}\setminus I_{k_1+1})\times(I_{k_2}\setminus I_{k_2+1})}S_{2^n,2^n}(x+s,y+t,|f|)\\&\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(s+t+e_{j_1}+e_{j_2})\right|d\mu(s,t)\\&\lesssim\sum_{j_1=0}^{n-1}\sum_{j_2=0}^{n-1}2^{j_1+j_2-2}\iint\limits_{I_{j_1}\times I_{j_2}}S_{2^n,2^n}(x+s,y+t,|f|)\\&\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(s+t+e_{j_1}+e_{j_2})\right|d\mu(s,t)\end{aligned}\tag{25}$$

$$=\sum_{j_1=0}^{n-1}\sum_{j_2=0}^{j_1}2^{j_1+j_2-2}\iint\limits_{I_{j_1}\times I_{j_2}}S_{2^n,2^n}\left(x+s,y+t,|f|\right)$$

$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(s+t+e_{j_1}+e_{j_2}\right)\right|d\mu\left(s,t\right)$$

$$+\sum_{j_1=0}^{n-1}\sum_{j_2=j_1+1}^{n-1}2^{j_1+j_2-2}\iint\limits_{I_{j_1}\times I_{j_2}}S_{2^n,2^n}\left(x+s,y+t,|f|\right)$$

$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(s+t+e_{j_1}+e_{j_2}\right)\right|d\mu\left(s,t\right)$$

$$=J_{11}+J_{12}.$$

It is easy to show that $s+t+e_{j_2}\in I_{j_2}$ for $s\in I_{j_1},t\in I_{j_2}$ and $j_2\leq j_1$. Hence, we can write

$$J_{11}\lesssim\sum_{j_1=0}^{n-1}\sum_{j_2=0}^{j_1}2^{j_1+j_2-2}\iint\limits_{I_{j_1}\times I_{j_2}}S_{2^n,2^n}\left(x+s,y+t+s+e_{j_2},|f|\right)\tag{26}$$

$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(t+e_{j_1}\right)\right|d\mu\left(s,t\right)$$

$$\lesssim2^{2n}\sum_{j_1=0}^{n-1}\sum_{j_2=0}^{j_1}2^{j_1+j_2-2}\iint\limits_{I_{j_1}\times I_{j_2}}\left(\iint\limits_{I_n\times I_n}\left|f\left(x+s+u,y+t+s+e_{j_2}+v\right)\right|d\mu\left(u,v\right)\right)$$

$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(t+e_{j_1}\right)\right|d\mu\left(s,t\right)$$

$$\lesssim2^{2n}\sum_{j_1=0}^{n-1}\sum_{j_2=0}^{j_1}2^{j_2-2}$$

$$\times\int\limits_{I_{j_2}}\left(\iint\limits_{I_n\times I_n}\left(2^{j_1}\int\limits_{I_{j_1}}\left|f\left(x+s+u,y+t+s+e_{j_2}+v\right)\right|d\mu\left(s\right)\right)\right)d\left(u,v\right)$$

$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}\left(x,y\right)w_m\left(t+e_{j_1}\right)\right|d\mu\left(t\right)$$

$$\lesssim\sum_{j_1=0}^{n-1}\sum_{j_2=0}^{j_1}2^{j_2-2}$$

$$\times \int\limits_{I_{j_2}} \left(2^n \int\limits_{I_n} \left(2^{j_1} \int\limits_{I_{j_1}} \left| f\left(x+s, y+t+s+e_{j_2}+v\right)\right| d\mu\left(s\right) \right) \right) d\left(v\right)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}\left(x,y\right) w_m\left(t+e_{j_1}\right) \right| d\mu\left(t\right).$$

Set

$$A_{j_1}\left(x,y\right) := 2^{j_1} \int\limits_{I_{j_1}} \left| f\left(x+s, y+s\right)\right| d\mu\left(s\right).$$

Then it is evident that

$$A_{j_1}\left(x,y+x\right) = 2^{j_1} \int\limits_{I_{j_1}} \left| f\left(x+s, y+x+s\right)\right| d\mu\left(s\right)$$

$$= 2^{j_1} \int\limits_{I_{j_1}} \left| F_2\left(x+s, y\right)\right| d\mu\left(s\right),$$

where

$$F_2\left(x,y\right) := f\left(x, y+x\right).$$

From the condition of the theorem it is evident that $F_2 \in L\log L\left(G\times G\right)$. On the other hand,

$$\sup_j A_j\left(x, x+y\right) \lesssim M_1 F_2\left(x,y\right).$$

Let

$$A\left(x,y\right) := \sup_j A_j\left(x,y\right).$$

It is evident that

$$\iint\limits_{G\times G} A\left(x,y\right) d\mu\left(x,y\right) = \iint\limits_{G\times G} A\left(x,y+x\right) d\mu\left(x,y\right) \tag{27}$$

$$\lesssim \iint\limits_{G\times G} M_1 F_2\left(x,y\right) d\mu\left(x,y\right)$$

$$\lesssim 1 + \iint\limits_{G\times G} \left|F_2\left(x,y\right)\right| \log^+ \left|F_2\left(x,y\right)\right| d\mu\left(x,y\right)$$

$$\lesssim 1 + \iint\limits_{G\times G} \left|f\left(x,y\right)\right| \log^+ \left|f\left(x,y\right)\right| d\mu\left(x,y\right).$$

Then from (26) we have

$$\left|J_{11}\right| \lesssim \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{j_1} 2^{j_2-2} \tag{28}$$

$$\times \int_{I_{j_2}} \left(2^n \int_{I_n} A\left(x, y+t+v+e_{j_2}\right) \right) d\left(v\right)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}\left(x,y\right) w_m\left(t+e_{j_1}\right) \right| d\mu\left(t\right)$$

$$\lesssim \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{j_1} 2^{j_2} \int_{I_{j_2}} S_{2^n}^{(2)}\left(x, y+t+e_{j_2}, A\right)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}\left(x,y\right) w_m\left(t+e_{j_1}\right) \right| d\mu\left(t\right)$$

$$\lesssim \sum_{j_1=0}^{n-1} \int_G \sum_{j_2=0}^{j_1} 2^{j_2} \mathbb{I}_{I_{j_2}}\left(t\right) S_{2^n}^{(2)}\left(x, y+t+e_{j_2}, A\right)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}\left(x,y\right) w_m\left(t+e_{j_1}\right) \right| d\mu\left(t\right)$$

$$\lesssim \sum_{j_1=0}^{n-1} \left(\int_G \left(\sum_{j_2=0}^{j_1} 2^{j_2} \mathbb{I}_{I_{j_2}}\left(t\right) S_{2^n}^{(2)}\left(x, y+t+e_{j_2}, A\right) \right)^p d\mu\left(t\right) \right)^{1/p}$$

$$\left(\int_G \left| \sum_{m=0}^{2^n-1} \alpha_{mn}\left(x,y\right) w_m\left(t+e_{j_1}\right) \right|^q d\mu\left(t\right) \right)^{1/q}.$$

Analogously as in (20) we can prove that

$$\left(\int_G \left| \sum_{m=0}^{2^n-1} \alpha_{mn}\left(x,y\right) w_m\left(t+e_{j_1}\right) \right|^q d\mu\left(t\right) \right)^{1/q}$$

$$\lesssim \left(\sum_{m=0}^{2^n-1} \left|\alpha_{mn}\left(x,y\right)\right|^p \right)^{1/p} \lesssim 1.$$

Hence

$$J_{11} \lesssim \sum_{j_1=0}^{n-1} 2^{j_1/q} V_2^{(p)}\left(x,y,A\right) \lesssim 2^{n/q} V_2^{(p)}\left(x,y,A\right),$$

where

$$A \in L_1\left(G \times G\right).$$

Analogously, we can prove that

$$J_{12} \lesssim 2^{n/q} V_1^{(p)}\left(x,y,A\right). \tag{29}$$

Combining (25), (28) and (29) we conclude that

$$|J_1| \lesssim 2^{n/q} V_1^{(p)}(x,y,A) + 2^{n/q} V_2^{(p)}(x,y,A). \tag{30}$$

Analogously, we can prove that

$$|J_2| + |J_4| \lesssim 2^{n/q} V_1^{(p)}(x,y,A) + 2^{n/q} V_2^{(p)}(x,y,A). \tag{31}$$

Combining (16), (17)-(24), (30),(31) we obtain of estimation (15).

Combining (10), (11), (12), (13), (15), (27) and Lemma 1 we conclude that

$$\begin{aligned} & \mu\{H_*^q f > \lambda\} \\ \lesssim\ & \frac{1}{\lambda}\left(\|M_1 f\|_1 + \|M_2 f\|_1 + \|A\|_1 + \lambda\mu\{Mf > \lambda\}\right) \\ \lesssim\ & \frac{1}{\lambda}\left(1 + \iint\limits_{G\times G} |f| \log^+ |f|\right). \end{aligned}$$

Theorem 1 is proved. □

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