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On a Rational Linear Approximation of Fourier Series for Smooth Functions

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The Fourier series of a smooth function on a compact interval usually has slow convergence due to the Gibbs phenomena. A class of Fourier-Pade approximations is introduced and studied for performing a boundary correction. Additional acceleration is achieved by applying Fourier–Bernoulli scheme.

KEY WORDS: Fourier series; pade approximation; rational approximation; L_2-Convergence.

1. INTRODUCTION

It is well known that for $f \in L_2(-1,1)$

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{i\pi n x}, \quad f_n = \frac{1}{2}\int_{-1}^{1} f(x) e^{-i\pi n x} dx \tag{1.1}$$

by L_2-norm

$$||f|| = \left(\int_{-1}^{1} |f(x)|^2 dx\right)^{1/2}.$$

For practical purposes, approximations are obtained by using only a finite number of Fourier coefficients $\{f_n\}$, $|n| \leqslant N$. As is well known [26], when we approximate f by truncated Fourier series (partial sum)

$$S_N(f) := \sum_{n=-N}^{N} f_n e^{i\pi n x}, \tag{1.2}$$

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the involved error is strongly dependent on the smoothness of f. Approximation of an 2-periodic $f \in C^{\infty}(R)$ function by S_N $(N \gg 1)$ is highly effective. When the approximated function has a point of discontinuity, this truncation procedure leads to the Gibbs phenomena. The "oscillations" caused by this phenomena are typically propagated into regions away from the singularity and degrade the quality of partial sum approximation.

Different ways of curing this deficiency have been suggested in the literature. Increasing the convergence rate of the Fourier series by subtracting a polynomial representing the discontinuities in the function and some of its derivatives was suggested by Krylov in 1933 [19] and later by Lanczos [20] (see also [2] with references). In Lanczos's work the polynomial is a linear combination of Bernoulli polynomials with "jumps" in coefficients. The key problem is the determination of the singularity amplitudes. As formulated by Gottlieb [14] the pointwise values of a piecewise smooth function can be found from its first N Fourier coefficients. This idea has been realized by Eckhoff in a series of papers [5–8]. There the "jumps" are determined by the corresponding system of linear equations. Further we shall refer to this approach as Krylov–Gottlieb–Eckhoff (KGE) method (see also [3,13,15,21,23,24] for multidimensional case). Exponential convergence has been derived in Geer and Banerjee [11] by utilization of trigonometric "basis" functions which have certain "built-in" singularities.

In a series of papers, Gottlieb and Shu [16–18] exploit the Gegenbauer polynomials and for analytic but not periodic function exponential convergence in the maximum norm was derived.

The idea to construct Pade approximations based on series representation of functions other than the classical power series [1] has been suggested and studied by several investigators. For example, Maehly [22] has suggested an approach to determine the coefficients in rational approximations based on Chebyshev series (see also [9]). The general form of Fourier-Pade representation has been suggested by Cheney [4], but he does not discuss any algorithm for the computation of coefficients, rates of convergence, etc Geer [12] introduced and studied a class of approximations to a periodic function f which uses the ideas of Pade, or rational approximations. Although these approximations do not "eliminate" the Gibbs phenomena, they do mitigate its effect. For elimination of the Gibbs phenomena an algorithm is described and studied in Driscoll and Fornberg [10] based on Pade-type approximations and utilizing the logarithmic function. The resulting "Singular Fourier–Pade" approximations are quite accurate anyway, except near the jumps.

We shall limit our discussion to functions which are smooth on $[-1, 1]$ with discontinuities only at the endpoints of the interval. A class of rational linear approximations $\{S_{p,N}(f)\}$, $p = 1, 2, \dots$; $N \to \infty$ based on ideas Fourier–Pade is introduced. Unknown parameters are determined by asymptotic L_2-errors from the corresponding minimization problems. Thus, the complexity of the resulting approximations are identical to Fourier partial sums with more efficient boundary correction compared to Fourier–Pade. The results are discussed in the context of Fourier–Pade approximations [12]. Additional acceleration was derived by combination with KGE method. The theory is further illustrated with several numerical examples.

2. Preliminaries

2.1. Main Approximation Formula

Consider a finite sequence of complex numbers $\theta := \{\theta_k\}_{|k|=1}^{p}$, $p \geqslant 1$ and denote

$$\Delta_n^0(\theta) = f_n,\ \Delta_n^k(\theta) = \Delta_n^{k-1}(\theta) + \theta_{k\,\mathrm{sgn}(n)} \Delta_{(|n|-1)\mathrm{sqn}(n)}^{k-1}(\theta),\ k \geqslant 1, \tag{2.1}$$

where $\mathrm{sgn}(n) = 1$ if $n \geqslant 0$ and $\mathrm{sgn}(n) = -1$ if $n < 0$.

From (1.1) and (1.2), we get

$$R_N(f) := f(x) - S_N(f) = R_N^+(f) + R_N^-(f),$$

$$R_N^+(f) := \sum_{n=N+1}^{\infty} f_n \mathrm{e}^{\mathrm{i}\pi n x}, \quad R_N^-(f) := \sum_{n=-\infty}^{-N-1} f_n \mathrm{e}^{\mathrm{i}\pi n x}.$$

It can easily be checked that for $|\theta_1| \neq 1$

$$R_N^+(f) = -\frac{\theta_1 f_N \,\mathrm{e}^{\mathrm{i}\pi(N+1)x}}{1 + \theta_1 \mathrm{e}^{\mathrm{i}\pi x}} + \frac{1}{1 + \theta_1 \mathrm{e}^{\mathrm{i}\pi x}} \sum_{n=N+1}^{\infty} \Delta_n^1(\theta) \mathrm{e}^{\mathrm{i}\pi n x}.$$

Reiteration of this transformation up to p times leads to the following expansion ($|\theta_k| \neq 1, k = 1, \dots, p$)

$$\begin{aligned} R_N^+(f) &= -\mathrm{e}^{\mathrm{i}\pi(N+1)x} \sum_{k=1}^{p} \frac{\theta_k \Delta_N^{k-1}(\theta)}{\prod_{s=1}^{k}(1 + \theta_s \mathrm{e}^{\mathrm{i}\pi x})} \\ &\quad + \frac{1}{\prod_{k=1}^{p}(1 + \theta_k \mathrm{e}^{\mathrm{i}\pi x})} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta) \mathrm{e}^{\mathrm{i}\pi n x}. \end{aligned}$$

Similar expansion of $R_N^-(f)$ reduces to the following approximation

$$S_{p,N}(f) := \sum_{n=-N}^{N} f_n \mathrm{e}^{\mathrm{i}\pi n x} - \mathrm{e}^{\mathrm{i}\pi(N+1)x} \sum_{k=1}^{p} \frac{\theta_k \Delta_N^{k-1}(\theta)}{\prod_{s=1}^{k}(1+\theta_s \mathrm{e}^{\mathrm{i}\pi x})}$$
$$-\mathrm{e}^{-\mathrm{i}\pi(N+1)x} \sum_{k=1}^{p} \frac{\theta_{-k} \Delta_{-N}^{k-1}(\theta)}{\prod_{s=1}^{k}(1+\theta_{-s} \mathrm{e}^{-\mathrm{i}\pi x})} \tag{2.2}$$

with an error

$$R_{p,N}(f) := f(x) - S_{p,N}(f) = R_{p,N}^+(f) + R_{p,N}^-(f), \tag{2.3}$$

where

$$R_{p,N}^{\pm}(f) := \frac{1}{\prod_{k=1}^{p}(1+\theta_{\pm k} e^{\pm i\pi x})} \sum_{n=N+1}^{\infty} \Delta_{\pm n}^{p}(\theta) \mathrm{e}^{\pm i\pi n x}. \tag{2.4}$$

If θ is the solution of a system

$$\Delta_n^p(\theta) = 0, \ n = -N-p, \dots, -N-1, N+1, \dots, N+p, \tag{2.5}$$

then approximation $S_{p,N}(f)$ coincides with Fourier-Pade approximation $[N+p/p]_f$ ([12]).

In this paper we introduce an alternative approach for determining the parameters θ_k by asymptotic L_2-errors (see Theorem 3.2) from the corresponding minimization problems (see Sec. 4.1 and Table I for optimal values of parameters in dependence of the smoothness of the approximated function for the cases $p = 1, 2, 3, 4$). We investigate these approach theoretically in the context of L_2-norm and illustrate the theory by numerical examples. In the last section we discuss the application of KGE approximation for additional acceleration of $S_{p,N}(f)$.

2.2. Auxillary Lemmas

In order to prove the main theorem, we need several lemmas.

Lemma 2.1. Let

$$\omega_{k,m} := \sum_{s=0}^{k} C_k^s (-1)^s s^m, \quad 0 \leqslant m \leqslant k, \tag{2.6}$$

where $C_k^s = \dfrac{k!}{s!(k-s)!}$; then

$$\omega_{k,m} = \begin{cases} 0, & m < k, \\ (-1)^k k!, & m = k. \end{cases} \tag{2.7}$$

Proof. Denote

$$\varphi_{k,0}(z) := (1+z)^k = \sum_{s=0}^{k} C_k^s z^s, \ \varphi_{k,m}(z) := z\,\varphi'_{k,m-1}(z), \ m \geqslant 1$$

and note that

$$\varphi_{k,m}(-1) = \omega_{k,m}.$$

The remaining is obvious. □

We say that $f \in C^q[-1,1]$ if $f^{(q)}$ is continuous in $[-1,1]$. By definition, put

$$A_k(f) = f^{(k)}(1) - f^{(k)}(-1), \ k = 0, \dots, q.$$

By $\gamma_k(p), k = 0, \dots, p$, we denote the coefficients of the polynomial

$$\prod_{k=1}^{p} (1 + \tau_k x) \equiv \sum_{k=0}^{p} \gamma_k(p) x^k. \tag{2.8}$$

Lemma 2.2. Suppose $f \in C^{q+p}[-1,1]$, $q \geqslant 0$, $p \geqslant 1$, $f^{(q+p+1)} \in L_1[-1,1]$ and $A_j(f) = 0$ for $j = 0, \dots, q-1$. If

$$\theta_k = \theta_{-k} = 1 - \frac{\tau_k}{N}, \quad k = 1, \dots, p, \tag{2.9}$$

then the asymptotic expansion holds as $N \to \infty$, $|n| \geqslant N+1$

$$\Delta_n^p(\theta) = A_q(f) \frac{(-1)^{n+p+1}}{2(i\pi)^{q+1} q!} \sum_{k=0}^{p} \frac{(q+p-k)!(-1)^k \gamma_k(p)}{N^k (n-k)^{q+1} |n-k|^{p-k}} + o(n^{-q-p-1}). \tag{2.10}$$

Proof. If in (2.1) $\theta_k = 1$, $|k| = 1, \dots, p$, we put $\Delta_n^k := \Delta_n^k(\theta)$. Notice that Δ_n^k are classical finite differences.

It is not hard to prove by induction that

$$\Delta_n^p(\theta) = \sum_{k=0}^{p} \frac{(-1)^k \gamma_k(p)}{N^k} \Delta_{n-k}^{p-k}. \tag{2.11}$$

Again by induction

$$\Delta_n^k = \sum_{j=0}^{k} C_k^j f_{(|n|-j)\mathrm{sgn}(n)}, \quad C_k^j = \frac{k!}{j!(k-j)!}. \tag{2.12}$$

Now we need asymptotic expansion of Fourier coefficients. By $p+q+1$-fold integration by parts in (1.1), we have the following

$$f_n = \frac{(-1)^{n+1}}{2} \sum_{s=q}^{p+q} \frac{A_s(f)}{(i\pi n)^{s+1}} + \frac{1}{2(i\pi n)^{p+q+1}} \int_{-1}^{1} f^{(p+q+1)}(x) \mathrm{e}^{-i\pi n x} dx, \tag{2.13}$$

where the second term is $o(n^{-p-q-1})$ as $n \to \infty$ according to the well-known Riemann–Lebesgue theorem ([26]). Combining this with (2.12), we get

$$\Delta_n^k = \frac{(-1)^{n+1}}{2} \sum_{s=q}^{p+q} \frac{A_s(f)}{(i\pi n)^{s+1}} \sum_{j=0}^{k} C_k^j \frac{(-1)^j}{\left(1 - \frac{j}{|n|}\right)^{s+1}} + o(n^{-p-q-1}) \tag{2.14}$$

as $n \to \infty$ and $0 \leqslant k \leqslant p$.

Then we put $g_s(x) = (1-x)^{-s-1}$ and proceed by Taylor expansion and Lemma 2.1

$$\begin{aligned} \sum_{j=0}^{k} C_k^j \frac{(-1)^j}{\left(1 - \frac{j}{|n|}\right)^{s+1}} &= \sum_{m=0}^{k} \frac{g_s^{(m)}(0)}{m!|n|^m} \omega_{k,m} + o(n^{-k}) \\ &= \frac{(k+s)!(-1)^k}{s!|n|^k} + o(n^{-k}), \quad n \to \infty. \end{aligned}$$

Substituting this in (2.14), we obtain

$$\Delta_n^k = A_q(f) \frac{(-1)^{n+k+1}(q+k)!}{2(i\pi n)^{q+1} q!\,|n|^k} + o(n^{-k-q-1}), \quad n \to \infty.$$

This together with (2.11) completes the proof. □

Lemma 2.3. Let τ_k, $k = 1, \ldots, p$ be complex numbers such that $\tau_i \neq \tau_j$ *for* $i \neq j$. Denote

$$b_m := \frac{(q+m-1)!}{q!} \sum_{j=1}^{p} \frac{\tau_j^{p-m}}{\prod_{\substack{i=1 \\ i \neq j}}^{p} (\tau_i - \tau_j)} \sum_{k=0}^{p-m} \frac{(-1)^{k+1} \gamma_k(p)}{\tau_j^k}, \tag{2.15}$$

$$1 \leqslant m \leqslant p, \quad q \geqslant 0, \ p \geqslant 1;$$

then

$$b_m = \begin{cases} (-1)^p, & m = 1 \\ 0, & m \geqslant 2. \end{cases} \tag{2.16}$$

Proof. Let $R > \max_{1 \leqslant k \leqslant p} |\tau_k|$. Evidently,

$$b_m = \frac{(q+m-1)!}{2\pi i q!} \sum_{k=0}^{p-m} (-1)^{k+1+p} \gamma_k(p) \int_{|\tau|=R} \frac{\tau^{p-m} d\tau}{\tau^k \prod_{i=1}^p (\tau_i - \tau)}. \tag{2.17}$$

In the case $m \geqslant 2$ the integral on the right-hand side of (2.17) tends to zero as $R \to \infty$ for all $0 \leqslant k \leqslant p-m$. This proofs the second part of (2.16).

In the case $m = 1$ the integral on the right-hand side of (2.17) tends to zero as $R \to \infty$ only for $k > 0$. If $k = 0$, we have

$$(-1)^p b_1 = -\frac{1}{2\pi i} \int_{|\tau|=R} \frac{\tau^{p-m} d\tau}{\prod_{i=1}^p (\tau_i - \tau)} = -\operatorname*{res}_{\tau=\infty} \psi_1(\tau) = 1.$$

This concludes the proof.

3. ASYMPTOTIC L_2-CONSTANTS

The following result is obvious from asymptotic expansion of f_n (see (2.13)).

Theorem 3.1. Suppose $f \in C^q[-1,1]$, $q \geqslant 0$, $f^{(q+1)} \in L_1[-1,1]$ and $A_j(f) = 0$ for $j = 0, \dots, q-1$; then the following estimate (see (1.2)) holds

$$\lim_{N \to \infty} N^{q+\frac{1}{2}} ||R_N(f)|| = |A_q(f)| c(q), \quad c(q) = \frac{1}{\pi^{q+1}\sqrt{2q+1}}. \tag{3.1}$$

Now we are interested in a similar result for $S_{p,N}(f)$.

Theorem 3.2. Suppose $f \in C^{q+p}[-1,1]$, $q \geqslant 0$, $p \geqslant 1$, $f^{(q+p+1)} \in L_1[-1,1]$ and $A_j(f) = 0$ for $j = 0, \dots, q-1$. If

$$\theta_k = \theta_{-k} = 1 - \frac{\tau_k}{N}, \quad k = 1, \dots, p, \quad \tau_k > 0, \ \tau_j \neq \tau_i, \ j \neq i; \tag{3.2}$$

then the following holds

$$\lim_{N \to \infty} N^{q+\frac{1}{2}} ||R_{p,N}(f)|| = |A_q(f)| c_p(q), \tag{3.3}$$

where

$$c_p(q) = \frac{1}{\pi^{q+1}} \left(\int_1^\infty \left| \phi_{p,q}(t) \right|^2 dt \right)^{1/2}, \qquad \phi_{p,q}(t) = \frac{(-1)^p}{t^{q+1}} \tag{3.4}$$

$$-\frac{1}{q!} \sum_{j=1}^{p} \frac{e^{-\tau_j(t-1)}}{\prod_{\substack{i=1 \\ i \neq j}}^{p} (\tau_i - \tau_j)} \sum_{k=0}^{p} \gamma_k(p)(-1)^{k+1} \sum_{m=0}^{p-k-1} (q+p-k-m-1)!\tau_j^m. \tag{3.5}$$

Proof. Note that for sufficiently large N we have $0 < \theta_j < 1$, $j = 1, \ldots, p$ and hence from (2.4) we derive

$$R_{p,N}^{+}(f) = \sum_{n=N+1}^{\infty} \Delta_n^p(\theta) \mathrm{e}^{\mathrm{i}\pi n x} \cdot \sum_{m=0}^{\infty} (-1)^m \mathrm{e}^{\mathrm{i}\pi m x} \sum_{k=1}^{p} \beta_k \theta_k^m, \tag{3.6}$$

where the numbers $\beta_j = \dfrac{N^{p-1}\theta_j^{p-1}}{\prod_{\substack{i=1 \\ i \neq j}}^{p} (\tau_i - \tau_j)}$ are defined by the identity

$$\frac{1}{\prod_{k=1}^{p} (1 + \theta_k \mathrm{e}^{\mathrm{i}\pi x})} \equiv \sum_{k=1}^{p} \frac{\beta_k}{1 + \theta_k \mathrm{e}^{\mathrm{i}\pi x}}.$$

From (3.6) we have

$$||R_{p,N}^{+}(f)||^2 = 2 \sum_{n=N+1}^{\infty} \left| \sum_{s=N+1}^{n} \Delta_s^p(\theta)(-1)^s \sum_{k=1}^{p} \beta_k \theta_k^{n-s} \right|^2$$

Taking into account Lemma 2.2, we derive as $N \to \infty$

$$N^{2q+1}||R_{p,N}^{+}(f)||^2 = \frac{|A_q(f)|^2}{2\pi^{2q+2}(q!)^2 N^3} \times \sum_{n=N+1}^{\infty} \left| \sum_{s=N+1}^{n} \sum_{m=0}^{p} \frac{(q+p-m)!(-1)^m \gamma_m(p)}{\left(\frac{s-m}{N}\right)^{q+p-m+1}} \sum_{k=1}^{p} \frac{\theta_k^{n-s+p-1}}{\prod_{\substack{i=1 \\ i \neq k}}^{p} (\tau_i - \tau_k)} \right|^2 + o(1). \tag{3.7}$$

Tending N to infinity and replacing the sums in (3.7) by corresponding integrals we get

$$\lim_{N\to\infty} N^{2q+1}||R^{+}_{p,N}(f)||^2 = \frac{|A_q(f)|^2}{2\pi^{2q+2}} \int_1^{\infty} |\psi(t)|^2\, dt, \tag{3.8}$$

where

$$\psi(t) := \sum_{k=1}^{p} \frac{1}{\prod_{\substack{i=1\\ i\neq k}}^{p}(\tau_i - \tau_k)} \sum_{m=0}^{p} \gamma_m I_{k,m}(t) \tag{3.9}$$

and

$$I_{k,m}(t) := (-1)^m \frac{(q+p-m)!}{q!} \int_1^t \frac{\mathrm{e}^{-\tau_k(t-x)}}{x^{q+p-m+1}} dx. \tag{3.10}$$

The passage from (3.7) to (3.8) is well-founded as integrand in (3.10) is continuous and enough decreasing as $t \to \infty$

$$\begin{aligned} |I_{k,m}(t)| &\leqslant \text{const}\left(\int_1^{t/2} \frac{\mathrm{e}^{-\tau_k(t-x)}}{x^{q+p-m+1}} dx + \int_{t/2}^{t} \frac{e^{-\tau_k(t-x)}}{x^{q+p-m+1}} dx\right) \\ &\leqslant \text{const}\left(e^{-\tau_k t/2} + t^{-1}\right). \end{aligned}$$

Integrating by parts, we derive

$$\begin{aligned} I_{k,m}(t) &= \frac{(-1)^{m+1}}{q!} \mathrm{e}^{-\tau_k(t-x)} \sum_{s=0}^{p-m-1} \tau_k^s \frac{(q+p-m-s-1)!}{x^{q+p-m-s}}\Bigg|_{x=1}^{x=t} \\ &\quad + (-1)^m \tau_k^{p-m} \int_1^t \frac{\mathrm{e}^{-\tau_k(t-x)}}{x^{q+1}} dx. \end{aligned}$$

According to definition of b_m (see Lemma 2.3) we have

$$\begin{aligned} \psi(t) &= \sum_{m=1}^{p} \frac{b_m}{t^{q+m}} \\ &\quad - \frac{1}{q!} \sum_{k=1}^{p} \frac{\mathrm{e}^{-\tau_k(t-1)}}{\prod_{\substack{i=1\\ i\neq j}}^{p}(\tau_i - \tau_j)} \sum_{m=0}^{p} \gamma_m (-1)^{m+1} \sum_{s=0}^{p-m-1} \tau_k^s (q+p-m-s-1)! \\ &\quad + \sum_{k=1}^{p} \frac{\tau_k^p}{\prod_{\substack{i=1\\ i\neq j}}^{p}(\tau_i - \tau_j)} \int_1^t \frac{\mathrm{e}^{-\tau_k(t-x)}}{x^{q+1}} dx \sum_{m=0}^{p} \frac{\gamma_m(-1)^m}{\tau_k^m}. \end{aligned} \tag{3.11}$$

The last term in (3.11) vanishes according to (2.8) and the first term can be simplified according to Lemma 2.3. This ends the proof as similar estimate is valid for $R^{-}_{p,N}(f)$. □

4. NUMERICAL RESULTS

4.1. Minimization of L_2-constants

In this section we solve the minimization problem of the errors $||R_{p,N}(f)||$, $p=1,\dots,4$ numerically for determining the unknown parameters τ_k in approximation $S_{p,N}(f)$ according to Theorem 3.2. In other words we minimize the integral on the right hand side of (3.4) by appropriate choice of parameters τ_k. Calculations are carried down by the global minimization possibilities of the package MATHEMATICA 5 ([25]) and the corresponding results are presented in Table I where the ratio $c(q)/c_p(q)$ describes effectiveness of optimal rational approximation $S_{p,N}(f)$ compared to $S_N(f)$. Further, under $S_{p,N}(f)$ we mean the approximation (2.2) with optimal choice of parameters τ_k as in Table I. We see that, for example when $q=3$, the approximations $S_{p,N}(f), p=1,\dots,4$ are more precise ($N \gg 1$) than the approximation $S_N(f)$ correspondingly $12, 82, 411$ and 1704 times.

Results in Table I are asymptotical ($N \gg 1$) and it is interesting to look at the numerical behavior of approximations for small and moderate numbers N. It is also important to compare $S_{p,N}(f)$ with Fourier–Pade approximation. In Table II we represent L_2-errors by approximating the function $f(x)=(1-x^2)\sin(x-1)$. We see that for $N=32, 128, 512, 2048$ the approximation $S_{3,N}(f)$ is more precise than $S_N(f)$ correspondingly 5, 13, 33, 43, 61 times. The last number coincides with asymptotical estimate $\frac{c(1)}{c_3(1)}=61.3$ (see Table I). Notice also that $S_{3,N}(f)$ is 10 times more precise for $N=2048$ than Fourier–Pade approximation $[N/3]_f$.

Now we represent some numerical results in order to show how the considered approximations perform themselves from a pointwise point of view. In Fig. 1 we outline absolute errors at the point of $x=1$ while approximating the functions $f(x)=\sin(x-1)$ (a) and $f(x)=(1-x^2)^4\sin(x-1)$ (b) for $N=128$. These figures also well explain why the L_2-errors of approximation $S_{p,N}(f)$ are so small.

It is interesting that the situation is quite the contrary far from the end points of the interval $[-1,1]$. In Tables III and IV we show absolute errors in the interval $[0,0.1]$ while approximating the functions $f(x)=\sin(x-1)$ and $f(x)=(1-x^2)^4\sin(x-1)$ for $N=16$. It become obvious that the precision of Fourier–Pade approximation is the best in the regions far from the singularities.

Table I. Numerical Values of $c_p(q)$ and $c(q)/c_p(q)$ for $1 \leqslant q \leqslant 6$, $1 \leqslant p \leqslant 4$ Using the Numerical Optimal Values of Parameters τ_k, $1 \leqslant k \leqslant p$

q	1	2	3	4	5	6
$c_1(q)$	0.01009	0.00159	0.00031	0.00007	0.00001	$4 \cdot 10^{-6}$
$c(q)/c_1(q)$	5.7	9.0	12.2	15.5	18.7	22.0
τ_1	1.3533	2.3199	3.3020	4.2915	5.2845	6.2795
$c_2(q)$	0.00277	0.00030	0.00004	$8 \cdot 10^{-6}$	$1 \cdot 10^{-6}$	$3 \cdot 10^{-7}$
$c(q)/c_2(q)$	21.0	46.7	82.3	128.1	183.8	249.7
τ_1	2.7595	4.0837	5.3580	6.6001	7.8190	9.0202
τ_2	0.53199	1.1360	1.8177	2.5460	3.3060	4.0890
$c_3(q)$	0.00095	0.00007	9×10^{-6}	1×10^{-6}	2×10^{-7}	4×10^{-8}
$c(q)/c_3(q)$	61.3	185.1	411.6	771.8	1296.7	2017.4
τ_1	0.2510	0.6382	1.1230	1.6730	2.2699	2.9023
τ_2	1.28553	2.2362	3.2067	4.1868	5.1725	6.1617
τ_3	4.2225	5.7813	7.2573	8.6781	10.0589	11.4089
$c_4(q)$	0.00037	0.00002	2×10^{-6}	2×10^{-7}	4×10^{-8}	7×10^{-9}
$c(q)/c_4(q)$	156.5	621.3	1704.4	3794.9	7377.9	13034.6
τ_1	0.6663	0.3861	0.7379	1.1602	1.6358	2.1534
τ_2	0.1304	1.3458	2.0908	2.8748	3.6851	4.5146
τ_3	2.2056	3.4130	4.5975	5.7649	6.9188	8.0619
τ_4	5.7354	7.4661	9.0951	10.6547	12.1630	13.6315

Table II. L_2-errors While Approximating the Function $f(x) = (1 - x^2)\sin(x - 1)$

	N=8	N=32	N=128	N=512	N=2048
$S_N(f)$	0.003	0.0004	0.00005	6.5×10^{-6}	8×10^{-7}
$S_{2,N}(f)$	0.0006	0.00003	3.5×10^{-6}	4.4×10^{-7}	3.8×10^{-8}
$S_{3,N}(f)$	0.002	0.00003	1.5×10^{-6}	1.5×10^{-7}	1.3×10^{-8}
$[N/2]_f$	0.001	0.00009	0.00001	1.4×10^{-6}	1.3×10^{-7}
$[N/3]_f$	0.0009	0.00007	7.6×10^{-6}	6.6×10^{-7}	1.4×10^{-7}

Table III. Absolute Errors in the Interval $[0, 0.1]$ While Approximating the Function $f(x) = \sin(x - 1)$ for $N = 16$

	$S_N(f)$	$S_{2,N}(f)$	$S_{3,N}(f)$	$[N/2]_f$	$[N/3]_f$
Absolute error	0.008	7×10^{-6}	4×10^{-7}	6×10^{-8}	1×10^{-9}

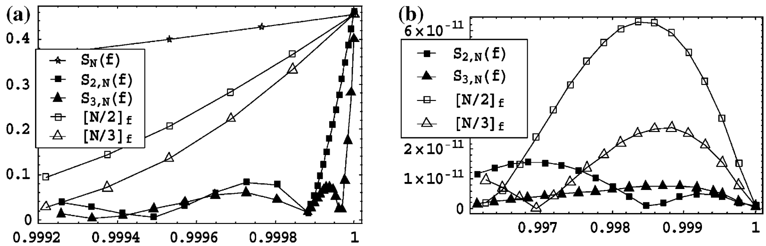

Fig. 1. Absolute errors at the point of $x = 1$ while approximating the functions $f(x) = \sin(x-1)$ (a) and $f(x) = (1-x^2)^4 \sin(x-1)$ (b) for N=128.

Table IV. Absolute Errors in the Interval $[0, 0.1]$ While Approximating the Function $f(x) = (1-x^2)^4 \sin(x-1)$ for $N = 16$

	$S_N(f)$	$S_{2,N}(f)$	$S_{3,N}(f)$	$[N/2]_f$	$[N/3]_f$
Absolute error	4×10^{-7}	2×10^{-9}	4×10^{-10}	6×10^{-11}	4×10^{-12}

4.2. Application of KGE-Method

The results in Table I make it clear that effectiveness of the approximation $S_{p,N}(f)$ is directly connected with the smoothness of f (parameter q). Thus, if $A_k(f) \neq 0$ for a small k we need a smoothing procedure for the approximated function. From this point of view we consider KGE approximation. As we have mentioned above the idea of KGE method is to split the given function f into two parts

$$f(x) = F(x) + \sum_{k=0}^{q-1} A_k(f) B_k(x), \tag{4.1}$$

where F is a relatively smooth function and $B_k(x)$ are the 2-periodic Bernoulli polynomials with Fourier coefficients

$$B_{k,n} = \begin{cases} 0, & n = 0 \\ \frac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, & n = \pm 1, \pm 2, \ldots \end{cases} \tag{4.2}$$

Approximation of the function F by the truncated Fourier series leads to KGE approximation

$$E_{0,q,N}(f) = \sum_{n=-N}^{N} F_n e^{i\pi n x} + \sum_{k=0}^{q-1} A_k(f) B_k(x), \tag{4.3}$$

Table V. L_2-errors While Approximating (4.6) by $E_{p,q,256}(f)$, $p=0,\dots,3$

	q=4	q=5	q=6	q=7
$E_{0,q,N}$	1.7×10^{-9}	1×10^{-9}	6.6×10^{-11}	3×10^{-9}
$E_{1,q,N}$	1.9×10^{-10}	5.8×10^{-11}	6.2×10^{-11}	3×10^{-9}
$E_{2,q,N}$	4.5×10^{-11}	7.0×10^{-12}	6.2×10^{-11}	3×10^{-9}
$E_{3,q,N}$	2.1×10^{-11}	2.4×10^{-12}	6.2×10^{-11}	3×10^{-9}

where the coefficients F_n can be expressed by f_n and $B_{k,n}$ from (4.1).

In order to determine the approximate values $\tilde{A}_k$ for $A_k(f)$ the fact that the coefficients F_n asymptotically $(n\to\infty)$ decay faster than the coefficients $B_{k,n}$ (however for $k=0$) is used, and can therefore be discarded for large $|n|$. Hence, from (4.1) we derive

$$f_n=\sum_{k=0}^{q-1}\tilde{A}_k B_{k,n},\quad n=n_1,n_2,\dots,n_m,\;\; n_i\neq n_j,\;\; i\neq j. \tag{4.4}$$

where the q unknowns are $\tilde{A}_k$, $k=0,\dots,q-1$ and $n=O(N)\leqslant N,\ N\to\infty$.

We use the same idea and approximate F by $S_{p,N}(f)$ deriving a new class of approximations

$$E_{p,q,N}(f)=S_{p,N}(F)+\sum_{k=0}^{q-1}A_k(f)B_k(x),\quad p=1,2,\dots \tag{4.5}$$

As a typical example let's consider the function

$$f(x)=\sinh(2.3x-0.6)\sin(23.5x+0.8) \tag{4.6}$$

in the interval $[-1,1]$. In Table V we represent L_2-errors for approximation $E_{p,q,N}(f)$, $p=0,\dots,3$, $N=256$. We calculate $\tilde{A}_k$ from (4.4) with $m=q+3$, $n_1=N, n_2=-N, n_3=N-1, n_4=-N+1,\dots$ by the least square algorithm. Besides, for additional precision we reconstruct the values of $\tilde{A}_k$ for $k=0,\dots,q+2$ and then use only for $k=0,\dots,q-1$.

It is well known that for big values of q $(q\geqslant 8)$ and N KGE-method is unstable. Unfortunately this is true even if the exact values of $A_k(f)$ are known. Consequently, it is useless to increase q without any limit. The conclusion that can be derived from here and many other similar experiments is as follows: fix q $(4\leqslant q\leqslant 6)$ and increase p.

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