Russian Math. Surveys **75**:3 569–571 *Uspekhi Mat. Nauk* **75**:3 183–184
DOI: https://doi.org/10.1070/RM9946

BRIEF COMMUNICATIONS

A sharp estimate for the majorant norm of a rearranged trigonometric system

G. A. Karagulyan

For a given orthonormal system $\Phi=\{\phi_n, n=1,2,\dots\}\subset L^2(0,1)$, an increasing sequence $w(n)$ of positive numbers is called a Weyl multiplier for almost everywhere convergence, or a C-multiplier, if every series $\sum_{n=1}^{\infty} a_n\phi_n(x)$ with coefficients satisfying $\sum_{n=1}^{\infty} a_n^2 w(n)<\infty$ is almost everywhere convergent. If such series converge almost everywhere after any rearrangement of the terms, then $w(n)$ is said to be a Weyl multiplier for almost everywhere unconditional convergence, or a UC-multiplier. The classical Menshov–Rademacher theorem (see [1], [2]) asserts that the sequence $\log^2 n$ is a C-multiplier for any orthonormal system.

The study of UC-multipliers for classical orthonormal systems has a long history. Kolmogorov [3] was the first to observe that the sequence $w(n)\equiv 1$ is not a UC-multiplier for the trigonometric system. A proof of this result was later given by Zahorski [4]. Developing Zahorski's argument, Ul'yanov [5] showed that the condition

$$\sum_{n=1}^{\infty}\frac{1}{nw(n)}<\infty \tag{1}$$

is necessary and sufficient for $w(n)$ to be a UC-multiplier for the Haar system. In his famous survey of 1964, Ul'yanov [6] posed the problem of estimating the growth of UC-multipliers for the trigonometric system and the Walsh system (see [6], § 11). After a number of partial results, Bochkarev [7] and Nakata [8] independently showed that (1) is a necessary condition for a sequence $w(n)$ to be a UC-multiplier for the Walsh system; for the trigonometric system the most general result so far was obtained in 1992 by Galstyan [9], who proved that the condition $\sum_{n=1}^{\infty}\frac{1}{n\log\log n\cdot w(n)}<\infty$ (which is slightly weaker than (1)) is necessary for a sequence $w(n)$ to be a UC-multiplier for the trigonometric system. As for sufficient conditions, nothing better is known to date than what has been obtained for general orthonormal systems.

AMS 2020 *Mathematics Subject Classification*. Primary 42C05, 42C10, 42C20.

Note that almost everywhere convergence problems are closely related to norm estimates for majorants of partial sums of this or that orthogonal series. In particular, the proof of the Menshov–Rademacher theorem is based on the inequality

$$\left\| \max_{1\leqslant m\leqslant n} \left| \sum_{k=1}^{m} a_k\phi_k \right| \right\|_2 \lesssim \log n \cdot \left\| \sum_{k=1}^{n} a_k\phi_k \right\|_2, \tag{2}$$

which holds for any orthonormal system $\{\phi_n\}$. Moreover, Menshov [1] showed that the logarithmic order of the coefficient in (2) cannot be improved. Problems of estimating the norms of majorants of various orthogonal sums are closely related to many problems in mathematical analysis.

Let Σ_N be the family of all one-to-one maps of $\{1,\dots,N\}$ onto itself (permutations). For a given integer $N\geqslant 1$ and a $\sigma\in\Sigma_N$, consider the permutation operator $T_{\sigma,N}\colon L^2(\mathbb{T})\to L^2(\mathbb{T})$ defined by

$$T_{\sigma,N}(x,f) = \max_{1\leqslant m\leqslant N}\left| \sum_{k=1}^{m} c_{\sigma(k)}e^{2\pi i\sigma(k)x} \right|,$$

where the c_k are the Fourier coefficients of a function $f\in L^2$. Our main result is as follows.

Theorem 1. *For any integer $N>1$, there exists a permutation $\sigma\in\Sigma_N$ such that $\|T_{\sigma,N}\|_{L^2\to L^2}\sim\log N$.*

The upper estimate in this theorem follows from the Menshov–Rademacher inequality (2). From the lower estimate one can easily derive the estimate

$$\|T_{\sigma,N}\|_{L^2\to L^{2,\infty}} = \sup_{\lambda>0}\lambda(|\{|T_{\sigma,N}f(x)|>\lambda\}|)^{1/2} \gtrsim \sqrt{\log N}$$

for the weak L^2-norm, which holds for some $\sigma\in\Sigma_N$. Using the last estimate, we prove the following.

Corollary 1. *A sequence $w(n)\nearrow\infty$ of positive numbers satisfying $w(n)=o(\log n)$ is not a* C*-multiplier for some rearrangement of the trigonometric system.*

Corollary 2. *The condition* (1) *is necessary for a sequence $w(n)$ to be a* UC*-multiplier for the trigonometric system.*

Problem 1. Does the inequality $\|T_{\sigma,N}\|_{L^2\to L^{2,\infty}}\lesssim\sqrt{\log N}$ hold for any rearrangement $\sigma\in\Sigma_N$?

We recall the following problem due to Kashin, which has become even more interesting after the estimate in our Theorem 1.

Problem 2 (Kashin, see [10], p. 595). Does there exist a sequence $w(n)=o(\log n)$ such that

$$\left(\int_0^1\int_0^1 \max_{1\leqslant m\leqslant n}\left|\sum_{k=1}^{m}\phi_k(x)\phi_k(y)\right|^2 dx\,dy\right)^{1/2} \leqslant w(n)\sqrt{n}$$

for any orthonormal system $\{\phi_n\}$ on $(0,1)$?

The author is grateful to B. S. Kashin for his helpful discussions and valuable comments.

Bibliography

[1] D. E. Menchoff (Menshov), *Fund. Math.* **4** (1923), 82–105.

[2] H. Rademacher, *Math. Ann.* **87**:1-2 (1922), 112–138.

[3] A. Kolmogoroff and D. Menchoff, *Math. Z.* **26**:1 (1927), 432–441.

[4] Z. Zahorski, *C. R. Acad. Sci. Paris* **251** (1960), 501–503.

[5] П. Л. Ульянов, *Матем. сб.* **60(102)**:1 (1963), 39–62. [P. L. Ul'yanov, *Mat. Sb.* **60(102)**:1 (1963), 39–62.]

[6] П. Л. Ульянов, *УМН* **19**:1(115) (1964), 3–69; English transl., P. L. Ul'yanov, *Russian Math. Surveys* **19**:1 (1964), 1–62.

[7] С. В. Бочкарев, *Изв. АН СССР. Сер. матем.* **43**:5 (1979), 1025–1041; English transl., S. V. Bočkarev, *Math. USSR-Izv.* **15**:2 (1980), 259–275.

[8] S. Nakata, *Anal. Math.* **5**:3 (1979), 201–205.

[9] С. Ш. Галстян, *Докл. РАН* **323**:2 (1992), 216–218; English transl., S. Sh. Galstyan, *Russian Acad. Sci. Dokl. Math.* **45**:2 (1992), 286–289.

[10] B. Sendov et al. (eds.), *Constructive function theory'* 81 (Varna, June 1–5, 1981), Publ. House Bulgar. Acad. Sci., Sofia 1983, 598 pp.

Grigori A. Karagulyan
Institute of Mathematics
of the National Academy of Sciences
of the Republic of Armenia;
Yerevan State University
E-mail: g.karagulyan@ysu.am

Presented by A. G. Sergeev
Accepted 20/MAR/20
Translated by A. ALIMOV