Minimization of Errors of the Polynomial-Trigonometric Interpolation with Shifted Nodes*

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Abstract

The polynomial-trigonometric interpolation based on the Krylov approach for a smooth function given on [-1, 1] is defined on the union of m shifted each other uniform grids with the equal number of points.

The asymptotic errors of the interpolation in both uniform and L_2 metrics are investigated. It turned out that the corresponding errors can be minimized due to an optimal choice of the shift parameters. The study of asymptotic errors is based on the concept of the "limit function" proposed by Vallee-Poussin. In particular cases of unions of two and three uniform grids the limit functions are found explicitly and the optimal shift parameters are calculated using MATHEMATICA 4.1 computer system.

The parallel processing is investigated.

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Introduction

It is known (see [1]) that the approximation of a function $f(x) \in C[-1,1]$ by the N-partial sum of Fourier series as $N \to \infty$ results in the Gibbs phenomenon with the constant $C_{ort} = 0.089...$, and the "overshoot" of approximated values in a neighborhood of the point $x = 1$ equals $C_{ort}(f(1) - f(-1))$. The Gibbs phenomenon with the greater constant $C_{int} = 0.141...$ (see [2]) also appears in the classical trigonometric interpolation on the uniform grid $\{x_k\} = \left\{\frac{2k}{2N+1}\right\}$, $k = 0, \pm 1, ..., \pm N$, $N \to \infty$.
In the described cases if $f(1) \neq f(-1)$ the uniform convergence on the segment $[-1,1]$ is failed and the order of L_2-convergence on compacts inside the interval $(-1,1)$ is greater than that on the whole interval.
However, also in the case $f(1) = f(-1)$ the situation generally is similar. Namely, if $f(x) \in C^{p+1}[-1,1]$, $p \geq 0$, $f^{(k)}(-1) = f^{(k)}(1)$, $k = 0, 1, ..., p$ and $f^{(p+1)}(-1) \neq f^{(p+1)}(1)$

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then although the uniform convergence on $[-1,1]$ holds, its order on compacts inside $(-1,1)$ is higher (see [1]).
An idea of the more precise approximation of a piecewise smooth function $f(x)$ on $[-1,1]$ by means of its Fourier coefficients $\{f_n\}$, $n = 0, \pm 1, ..., \pm N$ was proposed by A. N. Krylov in 1933 (see [3]). In the last decades some practical approaches were developed in a number of papers (see [4 - 6] and references therein). Applying this approach to approximation of a function $f(x) \in C^{p+1}[-1,1]$, $p \geq 0$, first it is constructed the polynomial $P(x)$, such that $f_1^{(k)}(1) = f_1^{(k)}(-1)$, $k = 0, 1, ..., p$, where $f_1(x) = f(x) - P(x)$, and then the function f_1 is approximated by the partial sum of Fourier series (or by the corresponding interpolation). It results in much faster approximation of $f(x)$.
The efficiency of numerical realization of this scheme and its generalization on the multi-dimensional case are based on finding of jumps $f^{(k)}(1) - f^{(k)}(-1)$, $k = 0, 1, ..., p$ (or multi-dimensional analogs of jumps) directly via the Fourier (discrete Fourier) coefficients of a function f (see [4 - 7] and Section 2.1 below). This scheme allows to approximate the function f with the uniform error of order $O(N^{-p-1})$, $N \to \infty$, $p \geq 0$, where N is the number of terms of the truncated Fourier series or grid points of uniform interpolation, respectively.
In [12] the interpolation on the union of three uniform grids shifted each other is considered. The study of asymptotic behavior of the uniform error was based on the concept of "limit function", proposed by Vallee-Poussin in 1908 already ([9]) for description of Gibbs phenomenon (see also [10]). It turned out that under the disposition of these three uniform grids the corresponding Gibbs constant may be even smaller than C_{ort} for the classical case of Fourier series. The additional drop of the error was obtained while considering the so-called "quasiperiodic" interpolation.
In this paper the investigations started in [12] are continued. Results of [12] are included without proofs. The general case of the union of m uniform grids with the equal number of points is considered. In particular cases of two and three grids the behavior of limit function near the endpoints of interval is studied and the asymptotic L_2-errors are found. The optimization problem of investigated interpolation is posed in uniform as well as in L_2-metrics. Its solution is explicitly obtained in cases of unions of two and three grids. In the general case parallel processing is investigated.

1 Polynomial-orthogonal expansion

1.1 Auxiliary lemma

For $y \in R$ by $y^{\vdash}$ we denote $y^{\vdash} = y (\mathrm{mod}\, 2)$, $0 \leq y^{\vdash} < 2$.
The next result will be often used below.

Lemma 1 *Let $x \in \mathbf{C}$, $x \neq 0, \pm 1, \pm 2, ...$; $q \geq -1$ be an integer, $w \in R$ and $w^{\vdash} \neq 0$ if $q = -1$. Then*

$$\Phi_q(w,x) \stackrel{def}{=} \sum_{s=-\infty}^{\infty} \frac{e^{i\pi w s}}{(x+s)^{q+2}} = 2i\pi \, \mathrm{Res}_{\mathrm{z}=-\mathrm{x}} \frac{\mathrm{e}^{\mathrm{i}\pi \mathrm{w}^{\vdash}\mathrm{z}}}{(1-\mathrm{e}^{2\mathrm{i}\pi\mathrm{z}})(\mathrm{x}+\mathrm{z})^{\mathrm{q}+2}}. \tag{1}$$

Proof. Let C_N be a circle in the complex z-plane with the radius $r_N = (N + \frac{1}{2})$ $(N \geq 1$

is an integer) and center at $z=0$. Let $0 \le w < 2$. Consider the contour integral

$$J_N = \frac{1}{2i\pi}\int_{C_N} \frac{e^{i\pi wz}}{(1-e^{2i\pi z})(x+z)^{q+2}}\,dz.$$

The location of C_N implies that the integral I_N exists and according to residue theory we have

$$J_N = -\frac{1}{2i\pi}\sum_{s=-N}^{N}\frac{e^{i\pi ws}}{(x+s)^{q+2}} + \mathrm{Res}_{z=-x}\frac{e^{i\pi wz}}{(1-e^{2i\pi z})(x+z)^{q+2}}.$$

For sufficiently large N and $q \ge -1$

$$|J_N| \le \frac{\mathrm{Const}}{N^{q+1}}\max_{|z|=r_N}\left|\frac{e^{i\pi(w-1)z}}{\sin(\pi z)}\right| \le \mathrm{Const}N^{-(q+1)}. \tag{2}$$

Hence $J_N \to 0$ as $N \to \infty$ if $q \ge 0$.
Now let $q=-1$, $w \ne 0$ and ε be a sufficiently small positive number. We divide the circle C_N into four arcs as follows: $C_N = c_+ \cup c_- \cup c_{up} \cup c_{down}$, where $c_+ = \{z : |z| = N+1/2, |\arg z| \le \varepsilon\}$, $c_- = \{z : |z| = N+1/2, \pi-\varepsilon \le \arg z \le \pi+\varepsilon\}$, $c_{up} = \{z : |z| = N+1/2,\ \varepsilon < \arg z < \pi-\varepsilon\}$, $c_{down} = \{z : |z| = N+1/2,\ \pi+\varepsilon < \arg z < 2\pi-\varepsilon\}$. The integral over $c_\varepsilon \cup c_{\pi-\varepsilon}$ can be made arbitrary small by choosing ε and taking into account an estimate of type (2).
Consider now the part of integral J_N over the upper arc c_{up}. Passing to the polar coordinates $z=(N+1/2)e^{i\phi}$ we have

$$\frac{(N+1/2)}{2\pi}\left|\int_\varepsilon^{\pi-\varepsilon}\frac{e^{i\phi}e^{\pi w(N+1/2)(i\cos\phi-\sin\phi)}}{(1-e^{2\pi(N+1/2)(i\cos\phi-\sin\phi)})(x+(N+1/2)e^{i\phi})}\,d\phi\right| \le$$
$$\frac{(N+1/2)}{2\pi}\int_\varepsilon^{\pi-\varepsilon}\frac{\left|e^{-\pi w(N+1/2)\sin\phi}\right|}{\left|1-e^{2\pi(N+1/2)(i\cos\phi-\sin\phi)}\right|\,\left|(N+1/2)-|x|\right|}\,d\phi \le$$
$$\mathrm{Const}\int_\varepsilon^{\pi-\varepsilon}\frac{\left|e^{-\pi w(N+1/2)\sin\phi}\right|}{\left|1-e^{2\pi(N+1/2)(i\cos\phi-\sin\phi)}\right|}\,d\phi$$

Since $0 < w < 2$ and $\sin\phi \ge \sin\varepsilon > 0$ on the segment $[\varepsilon, \pi-\varepsilon]$ for $N \to \infty$ the integrand can be estimated from above by Const $\left(e^{-w^{\vdash}(N+1/2)\sin\varepsilon}\right)$. Similarly, since $\sin\phi \le -\sin\varepsilon < 0$ on the segment $[\pi+\varepsilon, 2\pi-\varepsilon]$ the integrand can be estimated by Const $\left(e^{-(N+1/2)(2-w^{\vdash})\sin\varepsilon}\right)$.
It remains to note that by definition the function $\Phi_q(w,x)$ is 2-periodic relatively to w so we can replace w by $w^{\vdash}$ for any $w \in R$. ■

Remark 1 *It follows from formula (1) that if $w^{\vdash} \ne 0$ then $\Phi_q(w,t) \in C^\infty_{loc}$ as a function of w, while $\varphi_q(w,t) = \partial^{q+1}\Phi_q(w,t)/\partial w^{q+1}$ is a piecewise continuous function with jumps at $w^{\vdash}=0$. If ${w_0}^{\vdash}=0$ then the formally divergent series $\varphi_q(w_0,t)$ is summable in the sense of principal value (i.e. the summation in (1) is taken over symmetric limits $-A \le s \le A$, $A \to \infty$). It is not difficult to see that $\varphi_q(w_0,t) = (\varphi_q(w_0+0,t)+\varphi_q(w_0-0,t))/2$. Function $\Phi_q(w,t) - \frac{1}{t^{q+2}}$ is continuous in t for $t \in (-1,1)$ and an integer $q > -1$.*

1.2 Approximation via Fourier coefficients

First we describe the mentioned in Introduction method of restoration of a function $f(x) \in C^{q+1}[-1,1]$ by means of its Fourier coefficients

$$f_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-i\pi nx}\,dx, \quad n = 0, \pm 1, ..., \pm N, \quad N \geq 1. \tag{3}$$

The known scheme (see [3-5]) given here will be used to compare with the interpolation on shifted grids (see Section 6 below).
Let $\{B_k(x)\}_{k=0}^{\infty}$ be the Bernoulli polynomial system defined on $[-1,1]$ recurrently by the formulae:

$$B_0(x) = x/2, \quad B_k(x) = \int B_{k-1}(x)\,dx, \quad \int_{-1}^{1} B_k(x)\,dx = 0, \qquad k = 1, 2, ... \tag{4}$$

The Fourier coefficients $\{B_{kn}\}$ of a polynomial $B_k(x)$ have the form

$$B_{kn} = \begin{cases} 0, & n = 0 \\ \frac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, & n = \pm 1, \pm 2, ... \end{cases} \tag{5}$$

For a given integer N and q *polynomial-orthogonal expansion* of a function $f = f(x)$ is defined to be the following approximation

$$S_{N,q}(f) = \sum_{k=0}^{q} \mathcal{A}_k(f)B_k(x) + \sum_{n=-N}^{N}\left(f_n - \sum_{k=0}^{q}\mathcal{A}_k(f)B_{kn}\right)e^{i\pi nx}, \tag{6}$$

where $A_k(f)$ are the jumps of the function f and its derivatives at the endpoints of $[-1,1]$

$$\mathcal{A}_k(f) = f^{(k)}(1) - f^{(k)}(-1), \quad k = 0, 1, ..., q, \quad q \geq 0. \tag{7}$$

When $q = -1$ it is natural to deal with the partial sum of Fourier series

$$S_{N,-1}(f) = \sum_{n=-N}^{N} f_n e^{i\pi nx}. \tag{8}$$

Note that in practice it is not necessary to compute the jumps $\{\mathcal{A}_k(f)\}$ by (7). It is enough to find their approximate values.
The following known asymptotic representation of Fourier coefficients

$$f_n = \sum_{k=0}^{q+1} \mathcal{A}_k(f)B_{kn} + o\left(\frac{1}{n^{q+2}}\right), \quad n \to \infty \tag{9}$$

implies that the jumps $\{\mathcal{A}_k(f)\}$ can be restored with the precision $O(N^{-q+k-2})$, $k = 0, ..., q+1$, $N \to \infty$ by solving the system of equations with Vandermonde matrix

$$f_{n_s} = \frac{(-1)^{n_s+1}}{2}\sum_{k=0}^{q}\frac{\mathcal{A}_k(f)}{(i\pi n_s)^{k+1}}, \quad s = 0, 1, ..., q, \quad q \geq 0,$$

for various $(q+1)$ values of $\{n_s\}$ satisfying $const\,N \leq |n_s| \leq N$, $N \to \infty$ (see [4, 5]).

1.3 Asymptotic error

Let $x \in [-1,1]$. We are interested in the asymptotic behavior of the error

$$R_{N,q}(f)(x) = S_{N,q}(f)(x) - f(x) \tag{10}$$

for $q = \text{const}$, $N \to \infty$ provided that the exact values of the jumps (7) are known. It is known (see Introduction) that the maximum of this error is obtained, generally, near the endpoints of interval $[-1,1]$. Therefore, it is natural to consider the error $R_{N,q}(f)(x)$ at the point $x = 1 - \frac{h}{2N+1}$ (or $x = -1 + \frac{h}{2N+1}$) for $h = \text{const} > 0$ as $N \to \infty$.
Denote

$$Q_q(x) = -\cos(\pi(x + q/2)), \tag{11}$$

Consider now the following integral depending on a parameter h

$$E_q(h) = \frac{1}{\pi^{q+2}} \int_0^{\frac{1}{2}} \left(\frac{Q_q(ht)}{t^{q+2}} + \Psi_q(h,t) \right) dt, \tag{12}$$

where

$$\Psi_q(h,t) = \pi \text{Res}_{z=-t} \frac{e^{-i\pi z}((-1)^q e^{-i\pi ht} e^{i\pi h^{\vdash} z} + e^{i\pi ht} e^{i\pi (-h)^{\vdash} z})}{\sin(\pi z)(t+z)^{q+2}}. \tag{13}$$

The following results reveal the main term of asymptotic of (10) in uniform and L_2-metrics as $N \to \infty$.

Theorem 1 *([12]) Let $f(x) \in C^{q+2}[-1,1]$, $q \geq -1$. Then for $x = 1 - \frac{h}{2N+1}$ and $h = \text{const} > 0$*

$$\lim_{N\to\infty} (2N+1)^{q+1} R_{N,q}(f)(x) = \mathcal{A}_{q+1}(f) E_q(h). \tag{14}$$

Here and below we denote by $||f||$ the L_2-norm of a function $f \in L_2(-1,1)$.

Theorem 2 *Let $f(x) \in C^{q+2}[-1,1]$, $q \geq -1$. Then*

$$\lim_{N\to\infty} (2N+1)^{q+3/2} ||R_{N,q}(f)|| = \mathcal{A}_{q+1}(f) \frac{2^{q+3/2}}{\pi^{q+2}\sqrt{2q+3}}. \tag{15}$$

Proof. Let $f(x) = B_{q+1}(x)$. Using the orthogonality of Fourier system in $L_2[-1,1]$ we have for $N \to \infty$

$$||R_{N,q}(B_{q+1})||^2 = ||S_{N,q}(B_{q+1}) - B_{q+1}||^2 = 2 \sum_{|n| \geq N} \left| \frac{1}{2(i\pi n)^{q+2}} \right|^2 =$$

$$\frac{1}{(\pi(2N+1))^{2(q+2)}} \sum_{n \geq N} \frac{1}{(\frac{n}{2N+1})^{2(q+2)}}.$$

Now setting $t_n = \frac{n}{2N+1}$, $\Delta t = \frac{1}{2N+1}$ and rewriting the last sum we observe that this integral sum tends (as $N \to \infty$) to the following integral with the continuous integrand

$$\frac{1}{\pi^{2(q+2)}(2N+1)^{2q+3}} \int_{1/2}^{\infty} \frac{1}{t^{2(q+2)}} dt$$

From here it follows (15).
Let now $f(x)$ be an arbitrary function from C^{q+2}. We have from (9)

$$||R_{N,q}(f(x)) - \mathcal{A}_{q+1}(f) R_{N,q}(B_{q+1}(x))|| \leq$$
$$|| \sum_{|n| \geq N+1} (f_n - A_{q+1} B_{q+1,n}) e^{i\pi nx} || \leq o(1) (\sum_{|n| \geq N+1} n^{-2q-4})^{1/2} \leq o(1) N^{-q-3/2}. \tag{16}$$

■

2 Polynomial-trigonometric interpolation with shifted nodes

2.1 The problem

For a given α, $-1 < \alpha < 1$ we define the uniform grid $\{x_k^\alpha\} \in [-1, 1]$

$$x_k^\alpha = \frac{2k + \alpha + 1}{N} - 1, \qquad k = 0, ..., N - 1. \tag{17}$$

Denote by $\check{f}_n^\alpha$ the inverse discrete Fourier transform (up to a factor) of a function $f(x)$ on this grid

$$\check{f}_n^\alpha = \frac{1}{N} \sum_{k=0}^{N-1} f(x_k^\alpha) e^{-i\pi n x_k^\alpha}. \tag{18}$$

Here and below we denote

$$\sum_n^N \cdot = \sum_{n=-[N/2]}^{N-[N/2]-1} \cdot \quad .$$

Consider the following interpolation of a function f

$$I_N^\alpha(f)(x) = \sum_n^N \check{f}_n^\alpha a_n(x) \tag{19}$$

satisfying the conditions

$$I_N^\alpha(e^{i\pi px})(x) = e^{i\pi px}, \quad p = -[N/2], ..., N - [N/2] - 1, \quad x \in [-1, 1]. \tag{20}$$

According to the formula of discrete Fourier transform we have for a fixed p

$$I_N^\alpha(e^{i\pi px})(x) = \frac{1}{N} \sum_n^N \sum_{k=0}^{N-1} e^{i\pi(p-n)x_k^\alpha} a_n(x) = a_p(x).$$

From this it follows that $a_p(x) = e^{i\pi px}$. Hence we can rewrite (19) as

$$I_N^\alpha(f)(x) = \sum_n^N \check{f}_n^\alpha e^{i\pi nx}. \tag{21}$$

Let now $\bar{\alpha} = \{\alpha_1, ..., \alpha_m\}$ be a vector such that $-1 < \alpha_1 < ... < \alpha_m < 1$. We consider the union $\cup_{j=1}^m \{x_k^{\alpha_j}\}$ of the uniform grids consisting of an equal number N of nodes.
The aim of this paper is to minimize the interpolation errors by optimally disposing these grids, or, what is the same, by choosing an optimal shift parameter α.
Here we differ cases of odd and even m.
Let $\boldsymbol{m}$ **be odd**. Define the following interpolation

$$I_N^{\bar{\alpha}}(f)(x) = \sum_{j=1}^m a_j I_N^{\alpha_j} = \sum_{j=1}^m a_j \sum_n^N \check{f}_n^{\alpha_j} e^{i\pi nx} \tag{22}$$

satisfying the conditions

$$I_N^{\bar{\alpha}}(e^{i\pi px})(x) = e^{i\pi px}, \quad p = -[mN/2], ..., mN - [mN/2] - 1 \quad x \in [-1, 1]. \tag{23}$$

From (18) and (23) we obtain

$$\sum_{j=1}^{m} a_j \sum_{n}^{N} e^{i\pi(p-n)(\frac{\alpha_j+1}{N}-1)} e^{i\pi nx} \sum_{k=0}^{N-1} e^{\frac{2i\pi k(p-n)}{N}} = e^{i\pi px},$$

or representing p in the form $p = p_0 + sN$, $p_0 = -[N/2], ..., N - [N/2] - 1$, $s = -(m-1)/2, ..., (m-1)/2 - 1$,

$$\sum_{j=1}^{m} a_j \sum_{n}^{N} e^{i\pi(p_0-n+sN)(\frac{\alpha_j+1}{N}-1)} e^{i\pi nx} \sum_{s=-\infty}^{\infty} \delta_{n,p_0+sN} = e^{i\pi(p_0+sN)x},$$

where $\delta_{p,q}$ is the Kronecker symbol.

Thus for definition of a_j $(j = 1, ..., m)$ we obtain the following system of linear equations

$$\sum_{j=1}^{m} a_j e^{i\pi s\alpha_j} = (-1)^{s(N-1)} e^{i\pi sNx}, \quad s = -(m-1)/2, ..., (m-1)/2 - 1. \tag{24}$$

It is actually reduced to the linear system with the nonsingular (because $\alpha_i \neq \alpha_j (\mathrm{mod}\, 2)$ for $i \neq j$) Vandermonde matrix.

Let $\boldsymbol{m}$ **be even**. Define the following interpolation formula

$$I_N^{\bar{\alpha}}(f)(x) = \sum_{j=1}^{m} a_j I_N^{\alpha_j} = \sum_{j=1}^{m} a_j \sum_{n=0}^{N-1} \check{f}_n^{\alpha_j} e^{i\pi nx} \tag{25}$$

satisfying the conditions

$$I_N^{\bar{\alpha}}(e^{i\pi(p-mN/2)x})(x) = e^{i\pi(p-mN/2)x}, \quad p = 0, ..., mN-1 \quad x \in [-1, 1]. \tag{26}$$

Representing p in the form $p = p_1 + qN$, $p_1 = 0, ..., N-1$, $q = 0, ..., m-1$, from (18) and (26) we obtain

$$\sum_{j=1}^{m} a_j \sum_{n=0}^{N-1} (-1)^{p_1+N(q-m/2)-n} e^{i\pi(p_1+N(q-m/2)-n)\frac{(\alpha_j+1)}{N}} e^{i\pi nx} \sum_{s=-\infty}^{\infty} \delta_{n,p_1+N(q-m/2)} = e^{i\pi(p_1+N(q-m/2))x}. \tag{27}$$

For definition of a_j $(j = 1, ..., m)$ here we also obtain the system of linear equations with a nonsingular Vandermonde matrix

$$\sum_{j=1}^{m} e^{i\pi s\alpha_j} a_j = (-1)^{s(N-1)} e^{i\pi sNx}, \quad s = -m/2, ..., m/2 - 1. \tag{28}$$

As we see in both cases of even and odd m we have got for $\{a_j\}$ the same system.

2.2 Representation of the interpolation via Fourier coefficients

The next result shows that the coefficients of interpolation $\{\check{f}_n^{\alpha}\}$ can be expressed via Fourier coefficients of a function f.

Lemma 2 *Let $\{f_n\}$ be the Fourier coefficients of a function $f(x)$ and $|f_n| \le \text{const}\ n^{-p-1}$, $p > 0$. Then for $N \ge 1$ and $-1 \le \alpha \le 1$*

$$\check{f}_n^{\alpha} = \sum_{s=-\infty}^{\infty} f_{n+Ns} e^{i\pi(\alpha-N+1)s}. \tag{29}$$

Proof follows at once by substituting the absolutely and uniformly converging Fourier series of a function f into representation (18). ∎
The next result describes the error of the interpolation on the general uniform grid (17).

Lemma 3 *Let $\{f_n\}$ be the Fourier coefficients of a function $f(x)$ and $|f_n| \le \text{const}\ n^{-p-1}$, $p > 0$. Then the estimation*

$$\left| \sum_n^N \check{f}_n^{\alpha} e^{i\pi nx} - f(x) \right| \le \text{const}\, N^{-p}, \quad N \ge 1, \tag{30}$$

holds uniformly by $x \in [-1, 1]$.

Proof. From Lemma 2 we obtain

$$\sum_n^N \check{f}_n^{\alpha} e^{i\pi nx} - f(x) = \sum_n^N \left(\sum_{\substack{s=-\infty \\ s\ne 0}}^{\infty} f_{n+Ns} e^{i\pi(\alpha-N+1)s} \right) e^{i\pi nx} - \sum_{|n|>[N/2]} f_n e^{i\pi nx}.$$

The last term is of order $O(N^{-p})$ as $N \to \infty$. Taking into account that

$$\left| \sum_n^N \left(\sum_{\substack{s=-\infty \\ s\ne 0}}^{\infty} f_{n+Ns} e^{i\pi(\alpha-N+1)s} \right) e^{i\pi nx} \right| \le \frac{1}{N^{p+1}} \sum_n^N \sum_{\substack{s=-\infty \\ s\ne 0}}^{\infty} \frac{1}{|\frac{n}{N} + s|^{p+1}} \le \frac{2}{N^p} \sum_{s=1}^{\infty} \frac{1}{(s - \frac{1}{2})^{p+1}}.$$

we get (30). ∎

Corollary 1 *Under the assumptions of Lemma 3 the estimation*

$$|I_N^{\bar{\alpha}}(f)(x) - f(x)| \le \text{const} N^{-p}, \quad N \ge 1. \tag{31}$$

holds uniformly by x.

Proof follows from (22) (or (25)) and Lemma 3. ∎

2.3 Interpolation on the union of two uniform grids

Let consider the simplest case of union of two uniform grids symmetrically located with respect to origin.
Let $\bar{\alpha} = \{-\alpha, \alpha\}$ and $\{x_k^{\alpha}\}$, $0 < \alpha < 1$, be a uniform grid of the form (17). For $0 < \alpha < 1$ we consider the following interpolation of a function f on the grid of $2N$ points of the union $\{x_k^{-\alpha}\} \cup \{x_k^{\alpha}\}$

$$I_N^{\bar{\alpha}}(f)(x) = a_{-\alpha}(x) \sum_{n=0}^{N-1} \check{f}_n^{-\alpha} e^{i\pi nx} + a_{\alpha}(x) \sum_{n=0}^{N-1} \check{f}_n^{\alpha} e^{i\pi nx} \tag{32}$$

satisfying the conditions

$$I_N^{\bar{\alpha}}(e^{i\pi(p-N)x})(x) = e^{i\pi(p-N)x}, \quad p = 0, ..., 2N-1, \quad x \in [-1, 1]. \tag{33}$$

Note that for $\alpha = 1/2$ formula (32) corresponds to a trigonometric interpolation of a function $f(x)$ on a uniform grid with $2N$ nodes.
Similarly to (28) from (32) and (33) we get the following system of linear equations with respect to $\{a_{-\alpha}, a_\alpha\}$ ($x \in [-1, 1]$)

$$\begin{aligned}
&a_{-\alpha}(x) + a_\alpha(x) = 1, \\
&e^{i\pi\alpha} a_{-\alpha}(x) + e^{-i\pi\alpha} a_\alpha(x) = (-1)^{(N-1)} e^{-i\pi N x}.
\end{aligned}$$

As result for a fixed x we obtain

$$\begin{aligned}
a_{-\alpha}(x) &= -\frac{e^{-i\pi\alpha} + (-1)^N e^{-i\pi N x}}{2i\sin(\pi\alpha)}, \\
a_\alpha(x) &= \frac{e^{i\pi\alpha} + (-1)^N e^{-i\pi N x}}{2i\sin(\pi\alpha)}.
\end{aligned} \tag{34}$$

2.4 Interpolation on the union of three uniform grids

Here we consider the case of a union of three uniform grids symmetrically located with respect to origin.
Let $\bar{\alpha} = \{-\alpha, 0, \alpha\}$. For $0 < \alpha < 1$ consider the interpolation of a function f on the grid of $3N$ points of the union $\{x_k^{-\alpha}\} \cup \{x_k^0\} \cup \{x_k^\alpha\}$

$$I_N^{\bar{\alpha}}(f)(x) = a_{-\alpha}(x) \sum_n^N \check{f}_n^{-\alpha} e^{i\pi n x} + a_0(x) \sum_n^N \check{f}_n^0 e^{i\pi n x} + a_\alpha(x) \sum_n^N \check{f}_n^\alpha e^{i\pi n x} \tag{35}$$

satisfying the conditions

$$I_N^{\bar{\alpha}}(e^{i\pi p x})(x) = e^{i\pi p x}, \quad p = -[3N/2], ..., (3N - [3N/2] - 1), \quad x \in [-1, 1]. \tag{36}$$

As in (24) from (35), (36) we lead to the following system with respect to $\{a_{-\alpha}, a_0, a_\alpha\}$, $x \in [-1, 1]$

$$\begin{aligned}
&a_{-\alpha}(x) + a_0(x) + a_\alpha(x) = 1, \\
&e^{-i\pi\alpha} a_{-\alpha}(x) + a_0(x) + e^{i\pi\alpha} a_\alpha(x) = (-1)^{N-1} e^{i\pi N x}, \\
&e^{i\pi\alpha} a_{-\alpha}(x) + a_0(x) + e^{-i\pi\alpha} a_\alpha(x) = (-1)^{N-1} e^{-i\pi N x}.
\end{aligned}$$

Thus for a fixed x we obtain

$$\begin{aligned}
a_{-\alpha}(x) &= \frac{1}{4\sin^2\left(\frac{\pi\alpha}{2}\right)\cos\left(\frac{\pi\alpha}{2}\right)} \left(\cos\left(\frac{\pi\alpha}{2}\right) + (-1)^N \cos\left(\frac{\pi\alpha}{2} - \pi N x\right)\right), \\
a_0(x) &= -\frac{1}{2\sin^2\left(\frac{\pi\alpha}{2}\right)}((-1)^N \cos(\pi N x) + \cos(\pi\alpha)), \\
a_\alpha(x) &= \frac{1}{4\sin^2\left(\frac{\pi\alpha}{2}\right)\cos\left(\frac{\pi\alpha}{2}\right)} \left(\cos\left(\frac{\pi\alpha}{2}\right) + (-1)^N \cos\left(\frac{\pi\alpha}{2} + \pi N x\right)\right).
\end{aligned} \tag{37}$$

Note that for $\alpha = 2/3$ formula (35) is a trigonometric interpolation of a function $f(x)$ on a uniform grid with $3N$ nodes.

2.5 Total and quasiperiodic interpolations

We start from the general scheme of the classic pointwise interpolation of a function $f \in C[-1,1]$.
Let $N \geq 2$ be an integer, $\{x_k^N\} \subset [-1,1]$, $k = 1,2,...,N$ be a set of interpolation nodes and $\{T_n^N(x)\}$ be a given system of N linear independent functions, $T_n^N(x) \in C[-1,1]$, $n = 1,2,...,N$.
Consider the formula

$$I_N(f)(x) = \sum_{n=1}^{N} a_n T_n^N(x), \tag{38}$$

where a_n $(= a_n^N(f),\ n = 1,2,...,N)$ are defined to satisfy:

a) Formula (38) is interpolation on the grid $\{x_k^N\}$, i.e. if $f \in C[-1,1]$ then $I_N(f)(x_k^N) = f(x_k^N)$, $k = 1,2,...,N$;

b) Formula (38) is exact for the system $\{T_n^N(x)\}$, i.e. $I_N(T_n^N(x)) = T_n^N(x)$, $x \in [-1,1]$, $n = 1,2,...,N$.

Following to [8] let introduce two definitions.

Definition 1 *Interpolation (38) is called* **total** *if* $\min\{x_k\} = -1$, $\max\{x_k\} = 1$.

Show now that every not total interpolation generates the corresponding total one.
Assume that the interpolation (38) is not total, i.e. $b - a < 2$, where $a = \min_k\{x_k^N\}$, $b = max_k\{x_k^N\}$. Apply it to a function $f_1(x) = f((2x - a - b)/(b - a))$ that defined on the segment $[a,b]$ since $(2x - a - b)/(b - a) \in [-1,1]$ for $x \in [a,b]$. By inverse change of variable $x \to (b-a)x/2 + (a+b)/2$ we get the total interpolation formula

$$\tilde{I}(f)(x) = \sum_{n=1}^{N} a_n \widetilde{T}_n^N(x) \tag{39}$$

on the grid $\{(2x_k^N - a - b)/(b - a)\}$, $k = 1,2,...,N$, by means of the system $\{\widetilde{T}_n^N(x)\} = \{T_n^N((b-a)x/2 + (a+b)/2\}$, $n = 1,2,...,N$.

Definition 2 *Interpolation (38) is called* **quasiperiodic** *if it is defined for all $N \geq 1$, the system $\{T_n^N(x)\}$ is defined for $x \in R$, $T_n^N \in C_{\text{loc}}$, $T_n^N(x + t_N) = T_n^N(x)$, $t_N > 2$ and $t_N \to 2$ when $N \to \infty$.*

Observe that the periodic interpolation (i.e. $t_N = 2$) can not be total (see condition b)).
Though a quasiperiodic interpolation for $N \gg 1$ is close to a periodic one, in some cases it leads to the smaller errors than the periodic interpolation (see Section 6 below).
Applying the above-mentioned approach to the total quasiperiodic interpolation on the basis of formula (22) we note that its properties depend on a choice of a vector $\bar{\alpha}$. Below (see Sections 3 and 4) we consider the optimization problem of a polynomial-trigonometric interpolation in the sense of both uniform and L_2 convergences by means of choice of the parameter $\bar{\alpha}$ and application of the quasiperiodic interpolation.
This problem is explicitly solved below in two particular cases of the unions of two and three uniform grids.

3 Limit functions

3.1 Asymptotic uniform errors near the ends of interval

3.1.1 Union of two uniform grids

Assume that $\bar{\alpha} = \{-\alpha, \alpha\}$, $0 < \alpha < 1$. Let consider the union of two grids $\{x_k^{-\alpha}\} \cup \{x_k^{\alpha}\}$, $k = 0, ..., 2N-1$, consisting of $4N$ nodes (N is now replaced by $2N$ in formula (17)). Likewise (6), *polynomial-trigonometric interpolation* of a function f is defined to be the approximation formula

$$S_{N,q}^{\bar{\alpha}}(f)(x) = \sum_{k=0}^{q} A_k B_k(x) + I_N^{\bar{\alpha}} \left(f(x) - \sum_{k=0}^{q} A_k B_k(x) \right). \tag{40}$$

The result that follows characterizes the asymptotic behavior of the error

$$R_{N,q}^{\bar{\alpha}}(f)(x) = S_{N,q}^{\bar{\alpha}}(f)(x) - f(x) \tag{41}$$

in a neighborhood of the point $x = 1$.

Theorem 3 *Let $f(x) \in C^{q+2}[-1,1]$, $q \geq 0$, $\bar{\alpha} = \{-\alpha, \alpha\}$, $0 < \alpha < 1$. Then for $x = 1 - \frac{h}{4N}$ and $h = \mathrm{const} > 0$*

$$\lim_{N \to \infty} \left[(4N)^{q+1} R_{N,q}^{\bar{\alpha}}(f) \right] = \mathcal{A}_{q+1}(f) D_{q,\alpha}(h), \tag{42}$$

where

$$D_{q,\alpha}(h) = -\frac{2^q}{(i\pi)^{q+2}} \int_0^1 \left(e^{-\frac{i\pi h t}{2}} \mathrm{Res}_{\mathrm{z}=-\mathrm{t}} \left(\frac{\mathrm{e}^{-\mathrm{i}\pi \mathrm{z}} \mathrm{cosec}(\pi \mathrm{z})}{(\mathrm{t}+\mathrm{z})^{\mathrm{q}+2}} \left(\frac{\mathrm{i}}{2} \mathrm{cosec} \left(\pi\alpha \right) (\mathrm{e}^{\mathrm{i}\pi(\alpha+\mathrm{z}(-1+\alpha)^{\vdash}} - \right.\right.\right.$$
$$\left.\left.\left. e^{i\pi(-\alpha+z(-1-\alpha)^{\vdash}} \right) + e^{\frac{i\pi h}{2}} \left(e^{i\pi z(-1+\alpha)^{\vdash}} - e^{i\pi z(-1-\alpha)^{\vdash}} \right) \right) \right) \right) dt \tag{43}$$

Proof. Let $f(x) = B_{q+1}(x)$. According to (1), (5) and (29) ($n = 1, ..., 2N$, $q \geq 0$)

$$(\check{B}_{q+1}(x))_n^{\alpha} = \begin{cases} \frac{(-1)^{n+1} \Phi_q(\alpha+1, \frac{n}{2N})}{2(i\pi(2N))^{q+2}}, & n \neq 0 \\ \frac{1}{2} \sum_{\substack{s=-\infty \\ s \neq 0}}^{\infty} \frac{(-1)^{s+1}}{(2i\pi N s)^{q+2}}, & n = 0. \end{cases} \tag{44}$$

Using (34) let denote

$$u(\alpha, h) \stackrel{def}{=} a_{\alpha} \left(2N \left(1 - \frac{h}{4N} \right) \right) = \frac{e^{i\alpha\pi} + e^{\frac{i\pi h}{2}}}{2i \sin(\pi\alpha)}. \tag{45}$$

From the asymptotic expansion ($N \to \infty$, $x = 1 - \frac{h}{4N}$, $h = \mathrm{const} > 0$)

$$B_{q+1}(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^{n+1}}{2(i\pi n)^{q+2}} e^{i\pi n (1 - \frac{h}{4N})} =$$
$$-\frac{1}{2(i\pi)^{q+2}} \sum_{n=1}^{2N-1} \sum_{s=-\infty}^{\infty} \frac{e^{-\frac{i\pi(n+2Ns)h}{2N}}}{(n+2Ns)^{q+2}} - \frac{1}{2(i\pi)^{q+2}} \sum_{\substack{s=-\infty \\ s \neq 0}}^{\infty} \frac{e^{-\frac{i\pi h s}{2}}}{(2Ns)^{q+2}} =$$
$$-\frac{1}{2(i\pi(2N))^{q+2}} \sum_{n=1}^{2N-1} e^{-\frac{i\pi n h}{4N}} \sum_{s=\infty}^{\infty} \frac{e^{-\frac{i\pi h s}{2}}}{(\frac{n}{2N}+s)^{q+2}} + O\left(\frac{1}{N^{q+2}} \right) =$$
$$-\frac{1}{2(i\pi(2N))^{q+2}} \sum_{n=1}^{2N-1} e^{-\frac{i\pi n h}{4N}} \Phi_q \left(-\frac{h}{2}, \frac{n}{2N} \right) + O\left(\frac{1}{N^{q+2}} \right), \tag{46}$$

we get

$$R^{\bar{\alpha}}_{N,q}(B_{q+1}(x)) = S^{\bar{\alpha}}_{N,q}(B_{q+1}(x)) - B_{q+1}(x) =$$
$$\frac{1}{2(i\pi(2N))^{q+2}} \sum_{n=1}^{2N-1} e^{-\frac{i\pi nh}{4N}} \left(u(\alpha,h)\Phi_q\left(1+\alpha, \frac{n}{2N}\right) + \right.$$
$$\left. u(-\alpha,h)\Phi_q\left(1-\alpha, \frac{n}{2N}\right) - \Phi_q\left(-\frac{h}{2}, \frac{n}{2N}\right)\right) + O\left(\frac{1}{N^{q+2}}\right). \tag{47}$$

Denoting now $t_n = \frac{n}{2N}$, $\Delta t = \frac{1}{2N}$, we obtain

$$\lim_{N\to\infty}(4N)^{q+1} R^{\bar{\alpha}}_{N,q}(B_{q+1}(x)) = \frac{2^{q+1}}{2(i\pi)^{q+2}} \int_0^1 e^{\frac{-ih\pi t}{2}} (u(\alpha,h)\Phi_q(1+\alpha,t) +$$
$$u(-\alpha,h)\Phi_q(1-\alpha,t) - \Phi_q(-h/2,t))\, dt\,. \tag{48}$$

The passage to the limit is admissible since the integrand of the latter integral is smooth on $[0,1]$ (see Remark to Lemma 1). From here and (45) it follows (42).
Let now f be an arbitrary function from $C^{q+2}[-1,1]$. Denoting by $\omega(f,\varepsilon)$ the continuity modulus of function $f^{(q+2)}(x)$, from (9) we obtain (see [1], vol. 1)

$$|R_{N,q}(f(x)) - A_{q+1}(f) R_{N,q}(B_{q+1}(x))| \le \omega(f,1/N) \sum_{n=N}^{\infty} n^{-q-2} = o(N^{-q-1}).$$

This completes the proof. ■

3.1.2 Union of three uniform grids

Denote

$$\chi_q(u,w) = 2\pi \mathrm{Res}_{z=-t} \frac{\cos(\pi(q/2+u+(1-w)z))}{\sin(\pi z)(t+z)^{q+2}}. \tag{49}$$

The following theorem shows the asymptotic behavior of the error $R^{\bar{\alpha}}_{N,q}(f)$, $\bar{\alpha} = \{-\alpha, 0, \alpha\}$, for polynomial-trigonometric interpolation of a function f on the union of three grids $\{x_k^{-\alpha}\} \cup \{x_k^0\} \cup \{x_k^{\alpha}\}$, $k = 0, \pm 1, ..., \pm N$ in a neighborhood of the point $x = 1$.

Theorem 4 *([12]) Let $f(x) \in C^{q+2}[-1,1]$, $q \ge -1$, $\bar{\alpha} = \{-\alpha, 0, \alpha\}$. Then for $x = 1 - \frac{h}{6N+3}$ and $h = \mathrm{const} > 0$*

$$\lim_{N\to\infty} \left[(6N+3)^{q+1} R^{\bar{\alpha}}_{N,q}(f)\right] = \mathcal{A}_{q+1}(f) D_{q,\alpha}(h), \tag{50}$$

where

$$D_{q,\alpha} = \frac{3^{q+1}}{8\pi^{q+2}} \int_0^{\frac{1}{2}} \left(\csc^2\left(\frac{\pi\alpha}{2}\right) \left(\chi_q\left(\frac{ht}{3}, (1+\alpha)^{\vdash}\right) \left(\cos\left(\pi\left(\frac{\alpha}{2} - \frac{h}{3}\right)\right) \sec\left(\frac{\pi\alpha}{2}\right) + \right.\right.\right.$$
$$\left. +1\right) \chi_q\left(\frac{ht}{3}, (1-\alpha)^{\vdash}\right) \left(\cos\left(\pi\left(\frac{\alpha}{2} + \frac{h}{3}\right)\right) \sec\left(\frac{\pi\alpha}{2}\right) + 1\right) - 2i^{2q} \left(\cos\left(\frac{\pi h}{3}\right) + \right.$$
$$\left.\left. \cos(\pi\alpha)\right) \chi_q\left(-\frac{q}{2}, 1\right) \cos\left(\pi\left(\frac{q}{2} - \frac{ht}{3}\right)\right)\right) - 4\chi_q\left(\frac{ht}{3}, \left(-\frac{h}{3}\right)^{\vdash}\right)\right)\, dt. \tag{51}$$

4 L_2-asymptotic errors

In this section we are interested in the asymptotic behavior of L_2-norm of the error $R^{\bar{\alpha}}_{N,q}(f)$ in the interval $(-1,1)$ as $N \to \infty$.

4.1 Union of two uniform grids

Let consider the case of the union of two grids described in Section 3.1.1.

Theorem 5 *Let $f(x) \in C^{q+2}[-1,1]$, $q \ge -1$ and $\bar{\alpha} = \{-\alpha, \alpha\}$, $0 < \alpha < 1$. Then*

$$\lim_{N\to\infty} (4N)^{q+3/2} ||R^{\bar{\alpha}}_{N,q}(f)|| = \mathcal{A}_{q+1} V_{q,\alpha}, \tag{52}$$

where

$$V_{q,\alpha} = \frac{2^{q+1}}{\pi^{q+2}} \Bigg(\int_0^1 \left| -\frac{1}{(t-1)^{q+2}} - \frac{i\pi}{2} \mathrm{Res}_{\mathrm{z}=-\mathrm{t}} \frac{\mathrm{e}^{-\mathrm{i}\pi \mathrm{z}} \left(\mathrm{e}^{\mathrm{i}\pi\mathrm{z}(-1-\alpha)^{\vdash}} - \mathrm{e}^{\mathrm{i}\pi\mathrm{z}(-1+\alpha)^{\vdash}} \right)}{\sin(\pi\alpha)\sin(\pi \mathrm{z})(\mathrm{t}+\mathrm{z})^{\mathrm{q}+2}} \right|^2 +$$
$$\left| -\frac{1}{t^{q+2}} - \frac{i\pi}{2} \mathrm{Res}_{\mathrm{z}=-\mathrm{t}} \frac{\mathrm{e}^{-\mathrm{i}\pi\mathrm{z}} \left(\mathrm{e}^{\mathrm{i}\pi(-\alpha+\mathrm{z}(-1-\alpha)^{\vdash}}) - \mathrm{e}^{\mathrm{i}\pi(\alpha+\mathrm{z}(-1+\alpha)^{\vdash}}) \right)}{\sin(\pi\alpha)\sin(\pi\mathrm{z})(\mathrm{t}+\mathrm{z})^{\mathrm{q}+2}} \right|^2 -$$
$$\frac{1}{(t-1)^{2(q+2)}} - \frac{1}{t^{2(q+2)}} - \pi \mathrm{Res}_{\mathrm{z}=-\mathrm{t}} \frac{\mathrm{e}^{-\mathrm{i}\pi\mathrm{z}}}{\sin(\pi\mathrm{z})(\mathrm{t}+\mathrm{z})^{2(\mathrm{q}+2)}} \, \mathrm{dt} \Bigg)^{1/2}. \tag{53}$$

Proof. It is sufficient to prove the theorem for the function $f = B_{q+1}(x)$ (see 16)). We have the expansion

$$B_{q+1}(x) = \frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n+1}}{(i\pi n)^{q+2}} e^{i\pi nx} = \frac{1}{2} \sum_{n=1}^{2N-1} \sum_{s=-\infty}^{\infty} \frac{(-1)^{n+2Ns+1}}{(i\pi(n+2Ns))^{q+2}} e^{i\pi(n+2Ns)x} +$$
$$\frac{1}{2} \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \frac{(-1)^{2Ns+1}}{(i\pi 2Ns)^{q+2}} e^{i\pi 2Nsx} = \frac{1}{2} \sum_{n=1}^{2N-1} \sum_{s=-\infty}^{\infty} \frac{(-1)^{n+1} e^{i\pi(n+2Ns)x}}{(i\pi(n+2Ns))^{q+2}} + O\left(\frac{1}{N^{q+2}}\right). \tag{54}$$

Consider the L_2-norm of error (41) for $N \to \infty$ (see (1), (32), (40) and (44))

$$||R^{\bar{\alpha}}_{N,q}(B_{q+1})||^2 = \left|\left| S^{\alpha}_{N,q}(B_{q+1}(x)) - B_{q+1}(x) \right|\right|^2 =$$
$$\frac{1}{2(\pi(2N))^{2(q+2)}} \Bigg|\Bigg| \sum_{n=1}^{2N-1} (-1)^{n+1} e^{i\pi nx} \Bigg(a_\alpha(2Nx) \Phi_q \left(1+\alpha, \frac{n}{2N}\right) +$$
$$a_{-\alpha}(2Nx) \Phi_q \left(1-\alpha, \frac{n}{2N}\right) - \frac{1}{(\frac{n}{2N})^{q+2}} - \frac{e^{-i\pi(2N)x}}{(\frac{n}{2N}-1)^{q+2}} - \sum_{\substack{s=-\infty \\ s\neq -1,0}}^{\infty} \frac{e^{i\pi(2Ns)x}}{(\frac{n}{2N}+s)^{q+2}} \Bigg) +$$
$$O\left(\frac{1}{N^{2(q+2)}}\right) \Bigg|\Bigg|^2. \tag{55}$$

From the orthogonality of system $\{e^{i\pi nx}\}$, $n \in \mathbf{Z}$, in $L_2(-1,1)$ we obtain

$$\left|\left| \sum_{n=1}^{2N-1} \sum_{\substack{s=-\infty \\ s\neq -1,0}}^{\infty} \frac{(-1)^{n+1} e^{i\pi(n+2Ns)x)}}{(\frac{n}{2N}+s)^{q+2}} \right|\right|^2 = \Phi_{2q+2}\left(0, \frac{n}{2N}\right) - \frac{1}{(\frac{n}{2N})^{2(q+2)}} - \frac{1}{(\frac{n}{2N}-1)^{2(q+2)}}.$$

Collecting the terms by degrees of the exponent we have as $N \to \infty$

$$(4N)^{2q+3}||R^{\bar{\alpha}}_{N,q}(B_{q+1})||^2 = \frac{2^{2(q+1)}}{\pi^{2(q+2)}} \left(\sum_{n=1}^{2N-1} \left| -\frac{1}{(\frac{n}{2N}-1)^{q+2}} - \right.\right.$$
$$\left.\frac{i\pi}{2}\mathrm{Res}_{\mathrm{z}=-\frac{\mathrm{n}}{2\mathrm{N}}} \frac{\mathrm{e}^{-\mathrm{i}\pi\mathrm{z}}\left(\mathrm{e}^{\mathrm{i}\pi\mathrm{z}(1-\alpha)^{\vdash}} - \mathrm{e}^{\mathrm{i}\pi\mathrm{z}(1+\alpha)^{\vdash}}\right)}{\sin(\pi\alpha)\sin(\pi\mathrm{z})(\frac{\mathrm{n}}{2\mathrm{N}}+\mathrm{z})^{\mathrm{q}+2}} \right|^2 +$$
$$\left| -\frac{1}{\frac{n}{2N}^{q+2}} - \frac{i\pi}{2}\mathrm{Res}_{\mathrm{z}=-\frac{\mathrm{n}}{2\mathrm{N}}} \frac{\mathrm{e}^{-\mathrm{i}\pi\mathrm{z}}\left(\mathrm{e}^{\mathrm{i}\pi(-\alpha+\mathrm{z}(1-\alpha)^{\vdash}})-\mathrm{e}^{\mathrm{i}\pi(\alpha+\mathrm{z}(1+\alpha)^{\vdash}}\right)}{\sin(\pi\alpha)\sin(\pi\mathrm{z})(\frac{\mathrm{n}}{2\mathrm{N}}+\mathrm{z})^{\mathrm{q}+2}} \right|^2 -$$
$$\left. \frac{1}{(\frac{n}{2N}-1)^{2(q+2)}} - \frac{1}{\frac{n}{2N}^{2(q+2)}} - \pi\mathrm{Res}_{\mathrm{z}=-\frac{\mathrm{n}}{2\mathrm{N}}} \frac{\mathrm{e}^{-\mathrm{i}\pi\mathrm{z}}}{\sin(\pi\mathrm{z})(\frac{\mathrm{n}}{2\mathrm{N}}+\mathrm{z})^{2(\mathrm{q}+2)}} \right). \qquad (56)$$

Denoting now $t_n = \frac{n}{2N}$, $\Delta t = \frac{1}{2N}$, tending N to infinity and taking into account the smoothness of the integrand in (53) on $[0, 1]$ we obtain the statement of the theorem. ■

4.1.1 Minimization of the error of quasiperiodic interpolation

Apply now the scheme of Section 3.2 to the periodic (obviously not total) interpolation on $[-1, 1]$ by means of the system of periodic functions $\{T^N_n(x)\}$ which period greater than 2. If $\min\{x^N_k\} \to -1$ and $\max\{x^N_k\} \to 1$ for $N \to \infty$ we arrive at quasiperiodic interpolation.

For interpolation $I^{\bar{\alpha}}_N$, $\bar{\alpha} = \{-\alpha, \alpha\}$, $0 < \alpha < 1$ (see (32)) the total quasiperiodic interpolation is constructed by such parameters: $t_N = 2(1 + (1 - \alpha)/(2N + \alpha))$, and in (34) the argument x is replaced by $(\frac{2N+\alpha-1}{2N})x$.

The advantages of such a transfer from periodic interpolation to quasiperiodic one are obvious: in practice for the same algorithm the function is approximated exactly at the ends of the segment $[-1, 1]$ and in addition the error is essentially decreased. Moreover Gibbs phenomenon more naturally characterized at the ends of segment $[-1, 1]$ by both values of "overshoot" and "undershoot".

Show now that using the previous results we shall get the asymptotic formula of the total quasiperiodic interpolation. Indeed, denoting by $\varepsilon_{2,N,q}$ the error of the total quasiperiodic interpolation we obtain

$$||\varepsilon_{2,N,q}||^2 = \int_{-1}^{1} \left| R^{\bar{\alpha}}_{N,q}\left(\left(\frac{2N+\alpha-1}{2N}\right)x\right)\right|^2 dx = \frac{2N}{2N+\alpha-1}\int_{-1+\frac{1-\alpha}{2N}}^{1-\frac{1-\alpha}{2N}} |R^{\bar{\alpha}}_{N,q}(x)|^2\,dx =$$
$$\frac{2N}{2N+\alpha-1}\left(\int_{-1}^{1} |R^{\bar{\alpha}}_{N,q}(x)|^2\,dx - 2\int_{1-\frac{1-\alpha}{2N}}^{1} |R_{N,q}(x)|^2\,dx\right) =$$
$$\frac{2N}{2N+\alpha-1}\left(||R^{\bar{\alpha}}_{N,q}(x)||^2\,dx - d_{N,q}(\alpha)\right), \qquad (57)$$

where

$$d_{N,q}(\alpha) = \frac{1}{2N}\int_0^{2(1-\alpha)} \left| R^{\bar{\alpha}}_{N,q}\left(1 - \frac{h}{4N}\right)\right|^2 dh$$

is the value describing the improvement of asymptotic L_2-norm of error $R^{\bar{\alpha}}_{N,q}(x)$.

From here and Theorems 3 and 5 we get

Theorem 6 *Let $f(x) \in C^{q+2}[-1,1]$, $q \geq -1$, $\bar{\alpha} = \{-\alpha, \alpha\}$, $0 < \alpha < 1$, $D_{q,\alpha}$ and $V_{q,\alpha}$ be defined by (43) and (53) respectively Then*

$$\lim_{N\to\infty} (4N)^{q+3/2} ||\varepsilon_{2,N,q}|| = \mathcal{A}_{q+1} \sqrt{\left(V_{q,\alpha}^2 - 2\int_0^{2(1-\alpha)} |D_{q,\alpha}(h)|^2 \, dh \right)}. \tag{58}$$

Optimal choice of the parameter α corresponds to the minimization problem of the right-hand side of (58) if $\mathcal{A}_{q+1} = 1$.

4.2 Union of three uniform grids

4.2.1

For the union of three grids considered in item 3.1.2 the next result is true.

Theorem 7 *Let $f(x) \in C^{q+2}[-1,1]$, $q \geq -1$ and $\bar{\alpha} = \{-\alpha, 0, \alpha\}$. Then the asymptotic formula*

$$\lim_{N\to\infty} (6N+3)^{q+3/2} ||R_{N,q}^{\bar{\alpha}}(f)|| = A_{q+1} V_{q,\alpha}, \tag{59}$$

where

$$\begin{aligned} V_{q,\alpha} = & \frac{3^{q+3/2}}{\sqrt{2}\pi^{q+2}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\Phi_{2q+2}(0,t) - \frac{1}{t^{2(q+2)}} - \frac{1}{(t-1)^{2(q+2)}} - \frac{1}{(t+1)^{2(q+2)}} + \right.\right. \\ & \left| \frac{1}{(t-1)^{q+2}} + \frac{1}{4}\mathrm{cosec}^2\left(\frac{\alpha\pi}{2}\right) \left(\Phi_q(1,t) - 2\sec\left(\frac{\alpha\pi}{2}\right) \left(e^{\frac{i\pi\alpha}{2}} \Phi_q(1-\alpha,t) + \right.\right.\right. \\ & \left.\left.\left. e^{-\frac{i\pi\alpha}{2}} \Phi_q(1+\alpha,t) \right)\right)\right|^2 + \left| -\frac{1}{t^{q+2}} - \frac{1}{2}\mathrm{cosec}^2\left(\frac{\alpha\pi}{2}\right) (\cos(\alpha\pi)\Phi_q(1,t) - \right. \\ & \left. 2(\Phi_q(1-\alpha,t) + \Phi_q(1+\alpha,t)) \right|^2 + \left| \frac{1}{(t+1)^{q+2}} + \frac{1}{4}\mathrm{cosec}^2\left(\frac{\alpha\pi}{2}\right) (\Phi_q(1,t) - \right. \\ & \left.\left.\left. 2\sec\left(\frac{\alpha\pi}{2}\right) \left(e^{-\frac{i\pi\alpha}{2}} \Phi_q(1-\alpha,t) + e^{\frac{i\pi\alpha}{2}} \Phi_q(1+\alpha,t) \right)\right)\right|^2 \right) dt \right)^{1/2}. \end{aligned} \tag{60}$$

Proof is quite similar to the proof of Theorem 5.

4.2.2 Minimization of the error of quasiperiodic interpolation

For interpolation $I_N^{\bar{\alpha}}$, $0 < \alpha < 1$, (see (35)) the total quasiperiodic interpolation is constructed by the following parameters: $t_N = 2(1 + (1-\alpha)/(2N+\alpha))$, and in (37) the argument x is substituted on $(\frac{2N+\alpha}{2N+1})x$. Denoting by $\varepsilon_{3,N,q}$ the corresponding error of the total quasiperiodic interpolation, we have

$$\begin{aligned} ||\varepsilon_{3,N,q}||^2 = & \int_{-1}^{1} \left| R_{N,q}^{\bar{\alpha}} \left(\left(\frac{2N+\alpha}{2N+1} \right) x \right) \right|^2 dx = \frac{2N+1}{2N+\alpha} \int_{-1+\frac{1-\alpha}{2N}}^{1-\frac{1-\alpha}{2N}} |R_{N,q}^{\bar{\alpha}}(x)|^2 \, dx = \\ & \frac{2N+1}{2N+\alpha} \left(\int_{-1}^{1} |R_{N,q}^{\bar{\alpha}}(x)|^2 \, dx - 2\int_{1-\frac{1-\alpha}{2N}}^{1} |R_{N,q}^{\bar{\alpha}}(x)|^2 \, dx \right) = \\ & \frac{2N+1}{2N+\alpha} \left(||R_{N,q}^{\bar{\alpha}}(x)||^2 \, dx - d_{N,q}(\alpha) \right), \end{aligned} \tag{61}$$

where

$$d_{N,q}(\alpha) = \frac{2}{6N+3}\int_0^{3(1-\alpha)} \left| R_{N,q}^{\bar\alpha}\left(1 - \frac{h}{6N+3}\right)\right|^2 dh.$$

From here and Theorems 4 and 7 we get the next result.

Theorem 8 *Let $f(x) \in C^{q+2}[-1,1]$, $q \geq 0$ and $\bar\alpha = \{-\alpha, 0, \alpha\}$. Then*

$$\lim_{N\to\infty}(6N+3)^{q+3/2}||\varepsilon_{3,N,q}|| = \mathcal{A}_{q+1}\sqrt{\left(V_{q,\alpha}^2 - 2\int_0^{3(1-\alpha)} |D_{q,\alpha}(h)|^2\, dh\right)}, \qquad (62)$$

where $D_{q,\alpha}$ and $V_{q,\alpha}$ are defined by (51) and (60) respectively.

The problem is the minimization by α of the right-hand side of (62) if $\mathcal{A}_{q+1} = 1$.

5 Parallel processing

5.1

For simplicity here we consider the parallel processing problem only in the case of odd m. It will be easy to see that the suggested scheme is applicable also for even m.
The solution of system (24) has the following form

$$a_j = \sum_{j=1}^{m} \lambda_{js}(-1)^{Ns} e^{i\pi s N x}, \quad j = 1, 2, ..., m, \qquad (63)$$

where the matrix $\{\lambda_{js}\}$ is inverse to the Vandermonde matrix $\{e^{i\pi s\alpha_j}\}$, $j = 1, 2, ..., m$, $s = -(m-1)/2, ..., m-(m-1)/2-1$.
From here and (22) we have (see denotations in Section 1.3 above)

$$I_N^{\bar\alpha}(f)(x) = \sum_n^N F_{ns} e^{i\pi(n+Ns)x}, \qquad (64)$$

where

$$F_{ns} = (-1)^{Ns}\sum_{j=1}^{m}\sum_{s=-(m-1)/2}^{(m-1)/2} \lambda_{js}\check f_n^{\alpha_j},$$
$$n = -[N/2], ..., N-[N/2]-1,\ s = -(m-1)/2, ..., m-(m-1)/2-1. \qquad (65)$$

So to obtain the interpolation $I_N^{\bar\alpha}$ it is necessary:
a) to calculate the $N \times m$-matrix F_{ns},
b) to put out formula (64).
Pure numerical calculations are connected with item a). Let find its complexity taking into account all arithmetic operations. The coefficients $\{f_n^{\alpha_j}\}$ can be calculated by means of Fast Fourier Transform (FFT) whose complexity in the case of radix-2 scheme equals $6N\log_2 N + O(N)$ $(N \to \infty)$ operations for each j, $j = 1, ..., m$ (see [13]).
Calculation of the matrix $\{\lambda_{js}\}$ by the best at present Björck-Perejra algorithm for Vandermonde matrices (see [14]) requires $m^2 + O(m)$ operations.

Thus, for $m, N \gg 1$ the whole complexity Ω of calculation of matrix $\{F_{ns}\}$ is

$$\Omega = 6mN \log N + m^2 + O(\log N) + O(m).$$

Taking into account the form of (24) and (63) in case when $m \ll N$ it is reasonable to apply a multiprocessor system. Indeed, if we have m processors then each of them can calculate the coefficients $\{\check{f}_n^{\alpha_j}\}$ separately for the corresponding $j \quad (j = 1, 2, ..., m)$. Thus (see the previous item), the main part of operations of calculation $\{F_{nm}\}$ will be executed m times faster.
Let calculations be carried out on a cluster consisting of $(m+1)$ Computer Modules (CM): $\{\mathrm{CM}(1), ..., \mathrm{CM}(\mathrm{m}+1)\}$ provided each of them is actually a separate computer.
On Fig. 1 the chart of cluster working by the SIMD (Single Instruction Multiple Data) scheme is shown.
The function f and integer N are entered into input device (INPUT). Control Unit (CU) gives out the corresponding data $\{f(x_k^{\alpha_j})\}$ to each of the modules CM(j) $\quad (j = 1, ..., m)$ for treatment by the same program of FFT. These data are forwarded back to CU and sent to CM(m+1) for solving system (63), calculation of matrix $\{F_{ns}\}$ and symbolic writing of the formula produced at output device (OUTPUT).
It should be noted that connections among the modules $CM(1), ..., CM(m)$ are not activated here.

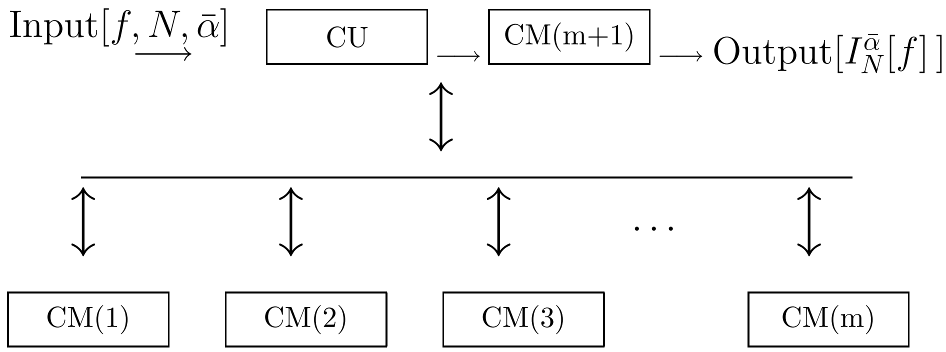

Fig. 1. *The scheme of calculating cluster for interpolation (*64*).*

It is not difficult to see that for this scheme the "speedup" efficiency coefficient without formation of the symbolic formula in $CM(m+1)$ equals $(N, m \gg 1)$ (see [16])

$$\text{speedup} = \frac{6N \log_2 N + m^2}{6mN \log_2 N + m^2}.$$

For $m = \text{const}$ and $N \gg 1$ speedup is maximal (practically equals $\frac{1}{m}$).
It is important that in the case of large m calculation of the matrix $\{\lambda_{js}\}$ may be evaluated by a parallel processing also because Björck-Perejra algorithm is vectorized. Here we will not stay on that.

5.2

Interpolation on the uniform grid with mN nodes corresponds to (22) if $\alpha_j = \frac{2j+\beta+1}{m}$ $(-1 < \beta = \text{const} < 1, \ j = 1, 2, ..., m)$. In this case the left side of (24) is discrete Fourier transform of $\{a_{\alpha_j}\}$ and $\{\lambda_{j_s}\}$ has an explicit form. For example, if m is odd it is easy to check that

$$I_N^{\bar{\alpha}}(f)(x) = \sum_{j=1}^{m} \frac{\sin\frac{\pi}{2}m(Nx - \frac{2j+1-\beta}{m})}{\sin\frac{\pi}{2}(Nx - \frac{2j+1-\beta}{m})} I_N^{\frac{2j+\beta+1}{m}}(f)(x). \tag{66}$$

So parallel processing mentioned above will be more simple.
On the other hand, each of FFT in modules CM(k) ($k = 1, ..., m$) of the scheme of Fig. 1 can be parallelized in its turn.
Indeed, if N allows the expansion $N = m_1 N_1$ then each of CM(k) on Fig. 1 may be replaced by a m_1-cluster with computer modules CM(k, r) ($r = 1, 2, ..., m_1$), each of which works by the similar scheme using (66) but instructions and data are get from the previous CU and results are output in CM(m+1). Applying this architecture we obtain that *speedup* equals $\frac{1}{mm_1}$.
At last note that for parallel processing of FFT in each of CM(k) ($k = 1, 2, ..., m$) one can use the other current multiprocessor system working by own scheme (see[17]). It is well-known many types of them. For example, it is possible to use the computation of FFT on the star-connected cycle network (see[18]) when the architecture is based on the n-star ($2 < n < N$) interconnected graph.

6 Numerical optimization

6.1

The presented results allow to obtain numerical values of asymptotic L_2-errors as well as the values of asymptotic uniform errors. However, here some calculating obstacles arise. First of all, it is connected with calculation of indeterminacies in integrands which are actually continuous. The direct application of an automatic integration package to the values E_q, G_q, $D_{q,\alpha}$ and V_{q_α} reduces (especially for large q) to the extreme accumulation of errors.
For example, in (43) the integrand (which is continuous on $[0, 1]$)contains indeterminacy of type $0/0$ at the points $x = 0$ and $x = 1$. To overcome these difficulties we have used the following approach: for a given sufficiently small parameter $\varepsilon > 0$ the interval is divided on three parts: $(0, \varepsilon)$, $(\varepsilon, 1 - \varepsilon)$, $(1 - \varepsilon, 1)$. On $(\varepsilon, 1 - \varepsilon)$ the integrand may be automatically integrated. As to intervals $(0, \varepsilon)$ and $(1 - \varepsilon)$ the function must be replaced in the neighborhood of the points $t = 0$ and $t = 1$ by the corresponding truncated Taylor series in the symbolic form. Note that we use MATHEMATICA 4.1 package possessed a powerful package for expansion of functions by the standard Taylor series formula. It turned out that with increasing q it is necessary to take more terms of Taylor series and to choose the appropriate ε to provide the desired precision.
In such way we have calculated the parts of the integrals (51), (53), (58), (60) and (62) containing the indeterminacies in a small neighborhoods of the ends of integration interval. It should be mentioned that this approach requires the computer with a high frequency and a big RAM. In our case PC Pentium III, 1GHz, 128 RAM was used.

6.2

Here we give numerical solutions of the problems of minimizations of both uniform and L_2 errors of interpolation (22) for some fixed q and $N \to \infty$ when considering both the periodic and quasiperiodic interpolations.
For convenience, we call *Gibbs Phenomenon value* (GPv) for given q and α the maximal value of $D_{q,\alpha}(h)$ (or $E_q(h)$ in the case of orthogonal decomposition) characterizing the asymptotic uniform error (see Theorems 1, 3 and 4). Correspondingly, we call the *value of L_2-error* for given α and q the value $V_{q,\alpha}$ characterizing the asymptotic L_2-error (Theorems 2, 5 and 7). Numerical results are given in Tables 1 - 3 up to 3 exact significant digits with the rounding last one.
Obviously the total quasiperiodic interpolation can not be applied to the polynomial-orthogonal decomposition. So for comparison the approximation errors are given for periodic (not total) interpolation. In Table 1 we present for the unions of 2 and 3 grids only GPvs, since we applied the total quasiperiodic interpolation and uniform error and GPv coincide for these cases.

q	OrthDec		UnGr	2 grids	3 grids
	UnErr	GPv	GPv	GPv	GPv
-1	0.500	0.0895	0.141	0.0834	0.0612
0	0.203	0.0706	0.0494	0.0223	0.0140
1	0.0495	0.0495	0.0453	0.0343	0.0308
2	0.0274	0.0187	0.0240	0.0356	0.0106
3	0.0119	0.0119	0.0115	0.0109	0.0101
4	0.00666	0.00551	0.00882	0.00630	0.00517

Table 1. *Uniform Error (UnErr) and GPv for the orthogonal decomposition (OrthDec), optimal GPv for uniform grid (UnGr) and unions of 2 and 3 grids.*

Results of Table 1 show that the suggested optimization is rather efficient. As for classic Gibbs Phenomenon value, the optimal values for 2 and 3 grids about twice less than in the case of the uniform grid. Note that the GPv for Fourier series equals 0.0895...
Table 2 shows that the asymptotic L_2-error is sufficiently improved when applying the total quasiperiodic interpolation. There are given the values of asymptotic L_2-errors corresponding to choice of the optimal parameter α in the cases of uniform, not total optimal, total uniform and total optimal grids.

q	OrthDec	UnGr	OptimGr		TotalUnGr	TotalOptimGr	
			2 Grids	3 Grids		2 Grids	3 Grids
-1	0.450	0.469	0.458	0.454	0.266	0.194	0.154
0	0.165	0.237	0.211	0.203	0.074	0.0441	0.0303
1	0.0816	0.107	0.103	0.0979	0.103	0.0957	0.0875
2	0.0439	0.0627	0.0611	0.0597	0.032	0.030	0.0239
3	0.0246	0.0345	0.0340	0.0336	0.0340	0.0339	0.0331
4	0.0142	0.0201	0.020	0.0199	0.0154	0.0136	0.0129

Table 2. *Values $V_{q,\alpha}$ in the cases of the orthogonal decomposition (OrthDec), uniform (UnGr) and total uniform grids (TotalUnGr), optimal (OptimGr) and total optimal grids (TotalOptimGr) for the unions of 2 and 3 grids.*

As we can see the suggested method is the best efficient in the case $q = 0$ when L_2-error decreases 5.4 times for 2 grids and 7.8 times for 3 grids.
It should be also mentioned that in case of the union of two and three grids corresponding to optimal α the significant decrease of errors takes place only for $-1 \leq q \leq 2$ and $q = 4$.

Below in Table 3 one can find the optimal values of the parameter α for which the asymptotic errors of uniform convergence (Gibbs phenomenon) and L_2-convergence of interpolations in the cases of not total and total grids, are the lowest. Thus, the grids corresponding to these values are optimal.

	α_{opt} for 2 grids			α_{opt} for 3 grids		
q	GPv	not Total (L_2)	Total (L_2)	GPv	not Total(L_2)	Total (L_2)
-1	0.407	0.432	0.329	0.550	0.607	0.487
0	0.345	0.647	0.294	0.469	0.781	0.436
1	0.436	0.436	0.416	0.590	0.593	0.567
2	0.380	0.553	0.310	0.504	0.721	0.480
3	0.486	0.471	0.470	0.631	0.624	0.619
4	0.411	0.519	0.394	0.537	0.692	0.540

Table 3. *Optimal values of the parameter α in the cases of the unions of 2 and 3 grids*

These results can be considered as instructions for practical applications.

References

[1] A. Zygmund, Trigonometric Series, vol. 1-2, Cambridge, 1959.

[2] G. Helmberg, *The Gibbs phenomenon for Fourier interpolation*, J. Approx. Theory, vol. 78, 1994, pp. 41 - 63.

[3] A.N. Krylov, Lectures on approximate calculations. [in Russian], Acad. of Sci. of USSR, Leningrad, 1933.

[4] D. Gottlieb, C.-W. Shu, A. Solomonoff, H. Vandeven, *On the Gibbs phenomenon I: recovering exponential accuracy from the Fourier partial sum of nonperiodic analytic function*, J. Comp. Appl. Math., vol. 43, 1992, pp. 81 - 92.

[5] K.S. Eckhoff, *On a high order numerical method for functions with singularities*, Math. of Comput., vol. 67, no. 223, 1998, pp. 1063 - 1087.

[6] G. Baszenski, F.J. Delvos, M. Tasche, *A united approach to accelerating trigonometric expansions.* Comp. Math. Appl., vol. 30, no. 3 - 6, 1995, pp. 33 - 49.

[7] A.B. Nersessian, A.V. Poghosyan, *Bernoulli method in multidimensional case.* Preprint no. 20 Ar- 00, 2000 deposited in ArmNIINTI 09.03.00, pp. 1 - 40.

[8] A.B. Nersessian, N.V. Hovhannisyan, *Quasiperiodic interpolation* [in Russian], Doklady NAN Armenii, vol. 101, no. 2, 2001, pp. 12 - 15.

[9] Ch. J. de la Vallee-Poussin, *Sur la convergence des formules d'interpolation entre ordonnées équidistantes*, Bull. Acad. Roy. Belgique, 1908, pp. 319 - 410.

[10] G. Helmberg, *A limit function for equidistant Fourier interpolation*, J. Approx. Theory, vol. 81, no. 3, 1995, pp. 389 - 396.

[11] N.S. Bakhvalov, N.P. Zhidkov, G.M. Kobel'kov, Numerical Methods [in Russian], Moscow, Nauka, 1987.

[12] A.B. Nersessian, N.V. Hovhannisyan, *Minimization of the uniform error of polynomial-trigonometric interpolation* [in Russian], Izvestiya NAN Armenii, to be published.

[13] H. Karner, M. Auer, Ch. Ueberhuber, *Multiply-Add Optimized FFT Kernel*, Math. Models and Methods in Appl. Sciences, vol. 11, no. 1, 2001, pp.105-117.

[14] A. Björck, V. Pereyra, *Solution of Vandermonde systems of equations*, Math. of Comput., vol. 24, no. 112, 1970, pp. 893 - 903.

[15] P.N. Swarztrauber, *The vector multiprocessors*, International Journal of High Speed Computing, vol. 11, no. 1, 2000, pp. 37 - 53.

[16] Y. Wallach, Alternating Sequential/Parallel Processing, Springer - Verlag, Berlin-Heidelberg-New York, 1982.

[17] R. Tolimieri, M. An, Ch. Lu, Algorithms for Discrete Fourier Transform and Convolution, Second Edition, Springer - Verlag, New York, 1997.

[18] S. Shoari, N. Bagherzadeh, *Computation of the fast Fourier transform on the star-connected cycle networks.* Computers Elect. Eng., vol. 22, no. 4, 1996, pp. 235 - 246.